Problem 12189
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Proposed by H. Katsuura (USA).

Evaluate

$$\int_0^1 \frac{(k+1)x^k - \sum_{j=0}^k x^{jk}}{x^{k(k+1)}-1}\,dx\,,$$

where k is a positive integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $0 < t < 1$, then

$$\begin{aligned}\int_0^t \frac{(k+1)x^k - \sum_{j=0}^k x^{jk}}{x^{k(k+1)}-1}\,dx &= \int_0^t \frac{1}{x^{k(k+1)}-1}\,d(x^{k+1}) - \int_0^t \frac{1}{x^k-1}\,dx \\ &= \int_0^{t^{k+1}} \frac{dx}{x^k-1} - \int_0^t \frac{dx}{x^k-1} = \int_{t^{k+1}}^t \frac{dx}{1-x^k} \\ &= \frac{1}{k}\left[-\frac{\ln(1-x^k)}{x^{k-1}}\right]_{t^{k+1}}^t + \frac{k-1}{k}\int_{t^{k+1}}^t \frac{-\ln(1-x^k)}{x^k}\,dx.\end{aligned}$$

Now, we note that

$$0 \le \frac{-\ln(1-x^k)}{x^k} \le -2^k\ln(1-x) \quad \forall x\in[1/2,1), \quad \text{and} \quad -\int_{1/2}^1 \ln(1-x) = \frac{1+\ln(2)}{2}$$

imply that $\int_{1/2}^1 \frac{-\ln(1-x^k)}{x^k}\,dx$ is finite and therefore

$$\lim_{t\to 1^-}\int_{t^{k+1}}^t \frac{-\ln(1-x^k)}{x^k}\,dx = 0.$$

Moreover, by letting $t = 1-s$ with $s \to 0^+$,

$$\begin{aligned}\left[-\frac{\ln(1-x^k)}{x^{k-1}}\right]_{t^{k+1}}^t &= -\frac{\ln(1-(1-s)^k)}{(1-s)^{k-1}} + \frac{\ln(1-(1-s)^{k(k+1)})}{(1-s)^{k^2-1}} \\ &= -\frac{\ln(ks+o(s))}{1+o(1)} + \frac{\ln(k(k+1)s+o(s))}{1+o(1)} \\ &= \left(-\ln(ks) + \ln(k(k+1)s)\right)(1+o(1)) \\ &= \ln(k+1) + o(1) \to \ln(k+1).\end{aligned}$$

Therefore, we may conclude that

$$\int_0^1 \frac{(k+1)x^k - \sum_{j=0}^k x^{jk}}{x^{k(k+1)}-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{(k+1)x^k - \sum_{j=0}^k x^{jk}}{x^{k(k+1)}-1}\,dx = \frac{\ln(k+1)}{k}.$$

□