Problem 12186
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Proposed by A. Eydelzon (USA).

For $v = \langle x_1, \ldots, x_n\rangle$ *in* $\mathbb{R}^n$, *let* $\|v\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ *and* $\|v\|_\infty = \max_{1\leq i\leq n}|x_i|$; *these are the usual* p*-norm and* ∞*-norm on* $\mathbb{R}^n$. *For what* v *does the series*

$$\sum_{p=1}^{\infty}(\|v\|_p - \|v\|_\infty)$$

converge?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We may assume that v is not the zero vector and $n > 1$, otherwise the series is trivially convergent. Then, we show that the series is convergent if and only if there is exactly one component of maximal absolute value.

(i) If the above condition is satisfied then, without loss of generality, let x_1 be the component of maximal absolute value and let $t = \frac{1}{|x_1|}\max_{2\leq i\leq n}|x_i| \in [0,1)$. Hence, as $p\to\infty$,

$$\begin{aligned}0 \leq \|v\|_p - \|v\|_\infty &\leq \|v\|_\infty\left((1+(n-1)t^p)^{1/p} - 1\right)\\ &= \|v\|_\infty\left(\exp\left(\frac{\ln(1+(n-1)t^p)}{p}\right) - 1\right) \sim \|v\|_\infty(n-1)\cdot\frac{t^p}{p}\end{aligned}$$

and the given series is convergent because $\sum_{p=1}^\infty \frac{t^p}{p} < \infty$.

(ii) If the above condition is not satisfied, then there are at least 2 components of maximal absolute value and therefore

$$\|v\|_p - \|v\|_\infty \geq \|v\|_\infty\left(2^{1/p} - 1\right) = \|v\|_\infty\left(\exp\left(\frac{\ln(2)}{p}\right) - 1\right) \sim \|v\|_\infty\ln(2)\cdot\frac{1}{p}$$

and the given series is not convergent because $\sum_{p=1}^\infty \frac{1}{p} = \infty$. $\square$