Problem 12184
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Prove

$$\int_1^\infty \frac{\ln(x^4-2x^2+2)}{x\sqrt{x^2-1}}\,dx = \pi\ln(2+\sqrt{2}).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. After letting $t=\sqrt{x^2-1}$, we have that

$$\int_1^\infty \frac{\ln(x^4-2x^2+2)}{x\sqrt{x^2-1}}\,dx = \int_0^\infty \frac{\ln(t^4+1)}{t^2+1}\,dt = I(\pi/4)+I(-\pi/4)$$

where

$$I(\alpha) := \int_0^\infty \frac{\ln(t^2+2\sin(\alpha)t+1)}{t^2+1}\,dt.$$

Now, for $\alpha\in(0,\pi/2)$,

$$\begin{aligned} I'(\alpha) &= \int_0^\infty \frac{2\cos(\alpha)t}{(t^2+1)(t^2+2\sin(\alpha)t+1)}\,dt \\ &= \frac{1}{\tan(\alpha)}\left(\int_0^\infty \frac{dt}{t^2+1} - \int_0^\infty \frac{dt}{t^2+2\sin(\alpha)t+1}\right) \\ &= \frac{\pi}{2\tan(\alpha)} - \frac{1}{\sin(\alpha)}\left(\frac{\pi}{2}-\alpha\right) = -\frac{\pi\tan(\alpha/2)}{2} + \frac{\alpha}{\sin(\alpha)}. \end{aligned}$$

Since

$$I(0) = \int_0^\infty \frac{\ln(t^2+1)}{t^2+1}\,dt = \int_0^{\pi/2} \frac{\ln(\tan^2(s)+1)}{\tan^2(s)+1}\,d(\tan(s)) = -2\int_0^{\pi/2} \ln(\cos(s))\,ds = \pi\ln(2)$$

it follows

$$I(\alpha) = I(0) + \int_0^\alpha I'(\alpha)\,d\alpha = \pi\ln\left(2\cos(\alpha/2)\right) + \int_0^\alpha \frac{a}{\sin(a)}\,da.$$

Hence

$$I(\alpha)+I(-\alpha) = \pi\ln\left(4\cos^2(\alpha/2)\right) = \pi\ln(2(1+\cos(\alpha)).$$

Finally, for $\alpha=\pi/4$ we find

$$I(\pi/4)+I(-\pi/4) = \pi\ln(2+2\cos(\pi/4)) = \pi\ln(2+\sqrt{2})$$

and we are done. □

Remark. Let $J=\int_0^{\pi/2}\ln(\cos(s))\,ds = \int_0^{\pi/2}\ln(\sin(s))\,ds$. Then

$$\begin{aligned} 2J &= \int_0^{\pi/2} \ln(\sin(s)\cos(s))\,ds = \int_0^{\pi/2} \ln(\sin(2s))\,ds - \int_0^{\pi/2} \ln(2)\,ds \\ &= \frac{1}{2}\int_0^{\pi} \ln(\sin(s))\,ds - \frac{\pi\ln(2)}{2} = J - \frac{\pi\ln(2)}{2} \end{aligned}$$

Hence $2J=-\pi\ln(2)$.