**Problem 12183**
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Proposed by H. Ohtsuka (Japan).

*For integers $m$, $n$, and $r$ with $m \geq 1$ and $n \geq r \geq 0$, prove*

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{q^{rm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1}$$

*where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the Gaussian binomial coefficient.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* We consider the partial fraction decomposition of the left-hand side:

$$\begin{aligned}
\frac{q^{rm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1} &= \frac{q^{rm} \prod_{j=1}^{n}(1-q^j)}{\prod_{j=0}^{n}(1-q^{j+m})} \\
&= \sum_{k=0}^{n} \frac{1}{1-q^{k+m}} \cdot \frac{q^{-rk} \prod_{j=1}^{n}(1-q^j)}{\prod_{j=0}^{k-1}(1-q^{j-k}) \prod_{j=k+1}^{n}(1-q^{j-k})} \\
&= \sum_{k=0}^{n} \frac{1}{1-q^{k+m}} \cdot \frac{q^{-rk} \prod_{j=1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^{-j}) \prod_{j=1}^{n-k}(1-q^{j})} \\
&= \sum_{k=0}^{n} \frac{1}{1-q^{k+m}} \cdot \frac{q^{-rk} \prod_{j=1}^{n}(1-q^j)}{(-1)^k q^{-\binom{k+1}{2}} \prod_{j=1}^{k}(1-q^{j}) \prod_{j=1}^{n-k}(1-q^{j})} \\
&= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q .
\end{aligned}$$

□