Problem 12182
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Proposed by G. Apostolopoulos (Greece).

Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Let R_A, R_B, and R_C denote the circumradii of triangles AEF, BFD, and CDE, respectively. Prove

$$R_A + R_B + R_C \le \frac{3R^2}{4r}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By the law of sines,

$$\frac{R_A}{R} = \frac{|EF|}{a}, \quad \frac{R_B}{R} = \frac{|FD|}{b}, \quad \frac{R_C}{R} = \frac{|DE|}{c}$$

where $a = |BC|$, $b = |CA|$, and $c = |AB|$. Moreover, by Cauchy-Schwarz inequality

$$\begin{aligned} R_A + R_B + R_C &= R\left(\frac{|EF|}{a} + \frac{|FD|}{b} + \frac{|DE|}{c}\right) \\ &\le R\sqrt{|EF|^2 + |FD|^2 + |DE|^2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}. \end{aligned}$$

Hence the required inequality follows as soon as we show that

$$\text{i)} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \le \frac{1}{4r^2} \quad \text{and} \quad \text{ii)} \quad |EF|^2 + |FD|^2 + |DE|^2 \le \frac{9R^2}{4}.$$

Let $a = x + y$, $b = y + z$, $c = z + x$ with $x, y, z > 0$. Then

$$s = x + y + z, \quad S^2 = xyz(x + y + z), \quad r = \frac{S}{s}, \quad R = \frac{abc}{4S}$$

where s is the semiperimeter of the triangle ABC and S is its area.

The inequality i) is

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \le \frac{x+y+z}{4xyz} = \frac{1}{4xy} + \frac{1}{4yz} + \frac{1}{4zx}$$

which holds because $4xy \le (x + y)^2$, $4yz \le (y + z)^2$, $4zx \le (z + x)^2$.

As regards ii), by the angle bisector theorem $CE = \frac{ab}{a+c}$ and $CD = \frac{ab}{b+c}$. Moreover, by the law of cosines,

$$\begin{aligned} |DE|^2 &= |CE|^2 + |CD|^2 - 2|CE| \cdot |CD| \cdot \cos(C) \\ &= \frac{a^2b^2}{(a+c)^2} + \frac{a^2b^2}{(b+c)^2} - \frac{ab(b^2 + a^2 - c^2)}{(a+c)(b+c)}. \end{aligned}$$

Hence we are able to write our inequality in terms of x, y, z, and we find that it is equivalent to

$$\begin{aligned} &36[8,1,0] + 288[7,2,0] + 288[7,1,1] + 945[6,3,0] + 2047[6,2,1]+ \\ &1647[5,4,0] + 3744[5,3,1] + 49[5,2,2] + 2237[4,4,1] \ge 7761[4,3,2] + 3520[3,3,3] \end{aligned}$$

where $[t_1, t_2, t_3] = \sum_{\text{sym}} x^{t_1} y^{t_2} z^{t_3}$, which holds by Muirhead's inequality. $\square$