Problem 12181
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Proposed by S. Sharma (India).

Prove

$$\sum_{k=2}^{\infty} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx = \gamma$$

where γ is the the Euler-Mascheroni constant.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that for all $x \in (0,1]$ there is a unique integer $m \geq 1$ such that

$$m \leq \frac{1}{\sqrt[k]{x}} < m+1 \Leftrightarrow \frac{1}{(m+1)^k} < x \leq \frac{1}{m^k}$$

and it follows that for $N \geq 2$

$$\begin{aligned}
\sum_{k=2}^{N} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx &= \sum_{k=2}^{N} \frac{1}{k} \int_0^1 \frac{1}{\sqrt[k]{x}}\, dx - \sum_{k=2}^{N} \frac{1}{k} \sum_{m=1}^{\infty} \int_{1/(m+1)^k}^{1/m^k} m\, dx \\
&= \sum_{k=2}^{N} \frac{1}{k-1} - \sum_{k=2}^{N} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}} + \frac{1}{(m+1)^k} \right) \\
&= \sum_{k=2}^{N} \frac{1}{k-1} - \sum_{k=2}^{N} \frac{1}{k} \left(1 + \sum_{m=1}^{\infty} \frac{1}{(m+1)^k} \right) \\
&= 1 - \frac{1}{N} - \sum_{k=2}^{N} \frac{1}{k} \sum_{m=2}^{\infty} \frac{1}{m^k}.
\end{aligned}$$

Therefore, as $N \to +\infty$,

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx &= 1 - \sum_{k=2}^{\infty} \frac{1}{k} \sum_{m=2}^{\infty} \frac{1}{m^k} \\
&= 1 - \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \frac{(1/m)^k}{k} \\
&= 1 + \sum_{m=2}^{\infty} \left(\frac{1}{m} + \ln\left(1 - \frac{1}{m}\right) \right) \\
&= 1 + \lim_{M \to \infty} \sum_{m=2}^{M} \left(\frac{1}{m} + \ln(m-1) - \ln(m) \right) \\
&= \lim_{M \to \infty} \left(\sum_{m=1}^{M} \frac{1}{m} - \ln(M) \right) = \gamma.
\end{aligned}$$

□