Problem 12180
(American Mathematical Monthly, Vol.127, April 2020)

Proposed by P. F. Refolio (Spain).

Prove

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} = \frac{2}{\pi} - \frac{\sqrt{2}\,C^2}{\pi^{3/2}} + \frac{\pi^{1/2}}{\sqrt{2}\,C^2},$$

where $C = \int_0^\infty t^{-1/4}e^{-t} = \Gamma(3/4)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since

$$\int_0^{\pi/2} \sin^{2n}(x)\,dx = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$$

and, for $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n} z^{2n}}{4^{2n}(2n+1)} = \frac{1}{z}\int_0^z \frac{f(z)+f(-z)}{2}\,dz = \frac{\sqrt{1+z}-\sqrt{1-z}}{z} = \frac{\sqrt{2}}{\sqrt{1+\sqrt{1-z^2}}}$$

with $f(z) = \sum_{n=0}^\infty \binom{2n}{n}\frac{z^n}{4^n} = \frac{1}{\sqrt{1-z}}$, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} &= \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)} \int_0^{\pi/2} \sin^{4n}(x)\,dx \\
&= \frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{dx}{\sqrt{1+\sqrt{1-\sin^4(x)}}} \qquad [\sin^2(x) = 2\sqrt{t(1-t)}] \\
&= \frac{1}{\sqrt{2}\pi}\int_0^{1/2} \frac{t^{1/2}+(1-t)^{1/2}}{t^{3/4}(1-t)^{5/4}}\,dt \\
&= \frac{1}{\sqrt{2}\pi}\left(\int_0^{1/2} t^{-1/4}(1-t)^{-5/4}\,dt + \int_0^{1/2} t^{-3/4}(1-t)^{-3/4}\,dt\right) \\
&= \frac{1}{\sqrt{2}\pi}\left(2\sqrt{2} - 2B_{1/2}(3/4,3/4) + B_{1/2}(1/4,1/4)\right)
\end{aligned}$$

where

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

is the so-called *Incomplete Beta Function* and we used the fact that

$$\frac{d}{dt}\left(4t^{3/4}(1-t)^{-1/4}\right) = t^{-1/4}(1-t)^{-5/4} + 2t^{-1/4}(1-t)^{-1/4}.$$

By noting that

$$B_{1/2}(a,b) = \frac{B_1(a,b)}{2} = \frac{\Gamma(a)\Gamma(b)}{2\Gamma(a+b)},$$

$\Gamma(3/2) = \Gamma(1/2)/2$ and, by the Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \frac{z}{\sin(\pi z)}$,

$$\Gamma(1/2) = \sqrt{\pi} \quad \text{and} \quad \Gamma(1/4)\Gamma(3/4) = \sqrt{2}\pi,$$

we finally find

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} &= \frac{1}{\sqrt{2}\pi}\left(2\sqrt{2} - \frac{\Gamma(3/4)^2}{\Gamma(3/2)} + \frac{\Gamma(1/4)^2}{2\Gamma(1/2)}\right) \\
&= \frac{2}{\pi} - \frac{\sqrt{2}\,\Gamma(3/4)^2}{\pi^{3/2}} + \frac{\pi^{1/2}}{\sqrt{2}\,\Gamma(3/4)^2}.
\end{aligned}$$

□