Problem 12179
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Proposed by N. MacKinnon (UK).

A positive integer n *is* good *if its prime factorization* $2^{a_1}3^{a_2}\cdots p_m^{a_m}$ *has the property that* a_i/a_{i+1} *is an integer whenever* $1 \le i < m$. *Find all* n *greater than* 2 *such that* $n!$ *is good.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $n > 2$, $n!$ is good if and only if $n \in \{3, 4, 5, 6, 7, 10, 11\}$.

By direct calculations we verify that the above claim holds for $n < 33$. For $n \ge 33$ we show that the prime factorization of $n!$ has a prime p with exponent 2 and a prime q with exponent 3. Since 2 does not divide 3 it follows that $n!$ is not good and the proof is complete.

By Jitsuro Nagura's paper *On the interval containing at least one prime number* (1952), for all $x \ge 25$ there is at least a prime in the interval $(x, 6x/5)$. Then it easily follows that for all $n \ge 33$ there is at least a prime $p \in (n/3, n/2]$ and there is at least a prime $q \in (n/4, n/3]$. Therefore, for $n \ge 33$,

$$11 \le \frac{n}{3} < p \le \frac{n}{2} \implies 2 \le \frac{n}{p} < 3 \quad \text{and} \quad \frac{n}{p^2} < \frac{n}{p(n/3)} < 1,$$

and

$$8 \le \frac{n}{4} < q \le \frac{n}{3} \implies 3 \le \frac{n}{q} < 4 \quad \text{and} \quad \frac{n}{q^2} < \frac{n}{q(n/4)} < 1.$$

Finally, by Legendre's formula,

$$\nu_p(n!) = \sum_{k=1}^{\lfloor \log_p(n) \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = 2 \quad \text{and} \quad \nu_q(n!) = \sum_{k=1}^{\lfloor \log_q(n) \rfloor} \left\lfloor \frac{n}{q^k} \right\rfloor = \left\lfloor \frac{n}{q} \right\rfloor = 3$$

where $\nu_p(m)$ denotes the exponent of the largest power of p that divides m. $\square$