Problem 12178
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Proposed by S. Portnoy (USA).

Given any function $f : \mathbb{R} \to \mathbb{R}$, show that there is a real number x and a sequence $x_1, x_2, \ldots$ of distinct real numbers such that $x_n \to x$ and $f(x_n) \to f(x)$ as $n \to \infty$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $\{q_i\}_{i\in\mathbb{N}^+}$ be an enumeration of $\mathbb{Q}$. For any $i, n \in \mathbb{N}^+$, let $J_{i,n}$ be the subset of $\mathbb{N}^+$ such that $j \in J_{i,n}$ if and only if

$$\left(q_i - \frac{1}{n}, q_i + \frac{1}{n}\right) \times \left(q_j - \frac{1}{n}, q_j + \frac{1}{n}\right) \cap \{(x, f(x) \ : \ x \in \mathbb{R}\} \neq \emptyset$$

i. e. the intersection of the open square centered at (q_i, q_j) and of side $2/n$, and the graph of f is non-empty.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that each set $J_{i,n}$ is non-empty. For any $i, n \in \mathbb{N}^+$ and any $j \in J_{i,n}$ let $x_{i,j,n} \in \mathbb{R}$ such that

$$(x_{i,j,n}, f(x_{i,j,n})) \in \left(q_i - \frac{1}{n}, q_i + \frac{1}{n}\right) \times \left(q_j - \frac{1}{n}, q_j + \frac{1}{n}\right),$$

and let X be the set of all those points $x_{i,j,n}$. It is evident that X is infinite-countable.
Let $x \in \mathbb{R} \setminus X$ (which is non-empty since $\mathbb{R}$ is uncountable). Then, by the density of $\mathbb{Q}$, there are two sequences $(q_{i_n})_n$ and $(q_{j_n})_n$ in $\mathbb{Q}$ such that

$$(x, f(x)) \in \left(q_{i_n} - \frac{1}{n}, q_{i_n} + \frac{1}{n}\right) \times \left(q_{j_n} - \frac{1}{n}, q_{j_n} + \frac{1}{n}\right).$$

Therefore $j_n \in J_{i_n,n}$ and, by the triangle inequality,

$$(x, f(x)) \in \left(x_{i_n,j_n,n} - \frac{2}{n}, x_{i_n,j_n,n} + \frac{1}{n}\right) \times \left(f(x_{i_n,j_n,n}) - \frac{2}{n}, f(x_{i_n,j_n,n}) + \frac{1}{n}\right).$$

Hence $x_{i_n,j_n,n} \to x$ and $f(x_{i_n,j_n,n}) \to f(x)$ as $n \to \infty$. Since $x \notin X$, there is a subsequence of $(x_{i_n,j_n,n})_n$ of distinct real numbers. □

Remark. Basically we used the fact that the metric space $(\mathbb{R}^2, d)$ where d is the Euclidean distance is separable ($\mathbb{Q}^2$ is countable and dense in $\mathbb{R}^2$) and therefore the metric subspace (G, d) is separable too where $G := \{(x, f(x)) \ : \ x \in \mathbb{R}\}$.