Problem 12175
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Proposed by G. Fera (Italy).

Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$2 < \frac{|AI|^2}{|AG|^2} + \frac{|BI|^2}{|BG|^2} + \frac{|CI|^2}{|CG|^2} \leq 3.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let m_a, m_b, m_c and w_a, w_b, w_c be, respectively, the lengths of the medians and the angle bisectors of triangle ABC. It is easy to verify that

$$|AI| = \frac{w_a(b+c)}{2s}, \qquad w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}$$
$$|AG| = \frac{2m_a}{3}, \qquad m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = s(s-a) + \frac{(b-c)^2}{4}.$$

Similar formulas hold for $|BI|$, w_b, $|BG|$, m_b, and $|CI|$, w_c, $|CG|$, m_c.

Upper bound. Since

$$m_a^2 \geq s(s-a), \quad m_b^2 \geq s(s-b), \quad m_c^2 \geq s(s-c),$$

it follows that

$$\frac{|AI|^2}{|AG|^2} + \frac{|BI|^2}{|BG|^2} + \frac{|CI|^2}{|CG|^2} = \frac{9bc(s-a)}{4sm_a^2} + \frac{9ca(s-b)}{4sm_b^2} + \frac{9ab(s-c)}{4sm_c^2}$$
$$\leq \frac{9(ab+bc+ca)}{(a+b+c)^2} \leq 3$$

where at the last step we used

$$3(ab+bc+ca) \leq (a+b+c)^2.$$

Lower bound. Since

$$m_a \leq \frac{b+c}{2}, \quad m_b \leq \frac{a+c}{2}, \quad m_c \leq \frac{a+b}{2},$$

Therefore

$$\frac{|AI|^2}{|AG|^2} + \frac{|BI|^2}{|BG|^2} + \frac{|CI|^2}{|CG|^2} = \frac{9bc(s-a)}{4sm_a^2} + \frac{9ca(s-b)}{4sm_b^2} + \frac{9ab(s-c)}{4sm_c^2}$$
$$\geq \frac{9bc(s-a)}{s(b+c)^2} + \frac{9ca(s-b)}{s(a+c)^2} + \frac{9ab(s-c)}{s(a+b)^2}$$
$$= \frac{9(w_a^2+w_b^2+w_c^2)}{4s^2} > 2$$

where at the last step we used

$$w_a^2 + w_b^2 + w_c^2 > \frac{8s^2}{9}$$

(see 11.7. at p. 218 in *Recent Advances in Geometric Inequalities* by Mitrinovic et al.).
For the sake of completeness we give a short proof of the last inequality: we have to show that

$$\frac{bc(s-a)}{(b+c)^2} + \frac{ca(s-b)}{(c+a)^2} + \frac{ab(s-c)}{(a+b)^2} > \frac{2s}{9}.$$

Let $a = x+y$, $b = y+z$, $c = z+x$ with $x, y, z > 0$, then the inequality is equivalent to

$$\sigma_1^3(\sigma_1^2 - 4\sigma_2)^2 + 6\sigma_1\sigma_3(\sigma_1\sigma_2 - 9\sigma_3) + \sigma_3(19\sigma_1^4 + 44\sigma_1^2\sigma_2 + 7\sigma_1\sigma_3 + 9\sigma_2^2) > 0$$

where $\sigma_1 = x+y+z$, $\sigma_2 = xy+yz+zx$, $\sigma_3 = xyz$. The above inequality holds because $\sigma_1\sigma_2 \geq 9\sigma_3$ by the AM-GM inequality, and $\sigma_i > 0$ for $i = 1, 2, 3$. □