Problem 12174
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(a) Let n be a positive integer, and suppose that the three leading digits of the decimal expansion of 4^n are the same as the three leading digits of 5^n. Find all possibilities for these three leading digits.
(b) Prove that for any positive integer k there exists a positive integer n such that the k leading digits of the decimal expansion of 4^n are the same as the k leading digits of 5^n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. (a) Let N be the integer given by the k common leading digits of 4^n and 5^n. Then $10^{k-1} \leq N < 10^k$ and there are $a, b \in \mathbb{N}^+$ such that

$$N \cdot 10^a < 4^n < (N+1) \cdot 10^a \quad \text{and} \quad N \cdot 10^b < 5^n < (N+1) \cdot 10^b.$$

Both strict inequalities for the lower bounds are due to the uniqueness of the prime-factorization. By multiplying the first inequality by the second one squared, we find

$$N^3 \cdot 10^{a+2b} < 100^n < (N+1)^3 \cdot 10^{a+2b}$$

which implies

$$10^{3(k-1)} \leq N^3 < 10^{2n-a-2b} < (N+1)^3 \leq 10^{3k}.$$

It follows that $2n - a - 2b \in \{3k-2, 3k-1\}$ and

$$N \in \{\lfloor 10^{k-2/3} \rfloor, \lfloor 10^{k-1/3} \rfloor\}.$$

Therefore, for $k = 3$ the possibilities for the three leading digits are $\lfloor 10^{7/3} \rfloor = 215$ and $\lfloor 10^{8/3} \rfloor = 464$. Both cases are attained

$$4^{712} = 464\ldots \quad , \quad 5^{712} = 464\ldots \quad \text{and} \quad 4^{2\cdot 712} = 215\ldots \quad , \quad 5^{2\cdot 712} = 215\ldots$$

(b) Since $\log_{10}(2)$ is an irrational number, it follows that the sequence $\{n \log_{10}(2) + 1/3 \pmod 1\}_{n \in \mathbb{N}}$ is dense in $[0, 1)$. Hence, for any $\varepsilon > 0$ there exist $n, m \in \mathbb{N}$ such that

$$m < n \log_{10}(2) + 1/3 < m + \log_{10}(1+\varepsilon),$$

that is

$$10^m < 2^n \cdot 10^{1/3} < 10^m(1+\varepsilon).$$

Therefore

$$10^{1/3} < \frac{4^n}{10^{2m-1}} < 10^{1/3}(1+\varepsilon)^2 \quad \text{and} \quad \frac{10^{1/3}}{1+\varepsilon} < \frac{5^n}{10^{n-m}} < 10^{1/3}.$$

By letting $\varepsilon = 1/10^k$, the above inequalities imply that that the first k leading digits of 4^n and 5^n are the same, i. e. $\lfloor 10^{k-2/3} \rfloor$. $\square$