Problem 12173
(American Mathematical Monthly, Vol.127, March 2020)

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Suppose that X and Y are n-by-n complex matrices such that $2Y^2 = XY - YX$ and the rank of $X - Y$ is 1. Prove $Y^3 = YXY$.

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Solution. Let $Z = (Y - X)/2$ then Z has rank 1 and there exist $v, w \in \mathbb{C}^n \setminus \{0\}$ such that $Z = vw^T$. Moreover, since $X = Y - 2Z$, it follows that

$$2Y^2 = XY - YX \Leftrightarrow Y^2 = YZ - ZY \quad \text{and} \quad Y^3 = YXY \Leftrightarrow YZY = 0.$$

Hence we have to prove that $YZY = 0$.

We first show that the eigenvalues of Y are all zero that is Y is nilpotent.
Let $\lambda_1, \dots, \lambda_r$ the pairwise distinct eigenvalues of Y and $m_1, \dots, m_r$ their multiplicities. For any integer $k \geq 0$

$$Y^{2+k} = Y^k YZ - Y^k ZY = Y(Y^k Z) - (Y^k Z)Y$$

and therefore

$$\sum_{i=1}^{r} t_i \lambda_i^k = \sum_{i=1}^{r} m_i \lambda_i^{2+k} = \text{tr}(Y^{2+k}) = \text{tr}(Y(Y^k Z)) - \text{tr}((Y^k Z)Y) = 0$$

where $t_i = m_i \lambda_i^2$ for $i = 1, \dots, r$. The (Vandermonde) determinant related to this homogeneous system of linear equations in the unknowns $t_1, \dots, t_r$ is nonzero because $\lambda_1, \dots, \lambda_r$ are pairwise distinct. Hence, it follows that $t_1 = \cdots = t_r = 0$, that is $r = 1$ and $\lambda_1 = 0$.

Finally we verify that $YZY = 0$. We have two cases.
1) If v is an eigenvector of Y then $Yv = 0$ and

$$YZY = Y(vw^T)Y = (Yv)(w^T Y) = 0.$$

2) If v is not an eigenvector of Y then v and Yv are linearly independent and the 2-dimensional vector space V generated by v and Yv is invariant with respect to Y:

$$Yv \in V \quad \text{and} \quad Y(Yv) = Y^2 v = YZv - ZYv = (w^T v)Yv - (w^T Yv)v \in V.$$

Since Y is nilpotent and V has dimension 2, by Cayley-Hamilton theorem, it follows that the restriction of Y^2 to V is identically zero and therefore

$$0 = Y^2 v = (w^T v)Yv - (w^T Yv)v \implies (w^T v) = 0 \quad \text{and} \quad (w^T Yv) = 0.$$

In particular $Z^2 = (vw^T)(vw^T) = v(w^T v)w^T = 0$. Therefore

$$0 = (Z+Y)(Y^2 - (YZ - ZY)) + (Y^2 - (YZ - ZY))(Z - Y) = 2YZY + Z^2 Y - YZ^2 = 2YZY + 0 + 0$$

which implies $YZY = 0$ and we are done. $\square$