Problem 12171
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Proposed by U. Abel and V. Kushnirevych (Germany).

Let T_n be the n-th Chebyshev polynomial, defined by $T_n(\cos(\theta)) = \cos(n\theta)$. Prove

$$\frac{T_n'(1/z)}{T_n(1/z)} = \frac{nz}{\sqrt{1-z^2}} + O(z^{2n+1})$$

as $z \to 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $\theta = it$, then $\cos(it) = \cosh(t)$ and we have that $T_n(\cosh(t)) = \cosh(nt)$. It follows that

$$\frac{T_n'(\cosh(t))}{T_n(\cosh(t))} = \frac{n\tanh(nt)}{\sinh(t)}.$$

In order to show the power series identity

$$\frac{T_n'(1/z)}{T_n(1/z)} - \frac{nz}{\sqrt{1-z^2}} = O(z^{2n+1}),$$

it suffices to show that

$$\lim_{z\to 0^+} \frac{1}{z^{2n+1}} \left(\frac{T_n'(1/z)}{T_n(1/z)} - \frac{nz}{\sqrt{1-z^2}}\right)$$

exists and it is finite. As $z \to 0^+$, $t := \operatorname{arccosh}(1/z) \to +\infty$, $\frac{\sqrt{1-z^2}}{z} = \sinh(t)$ and

$$\begin{aligned}
\lim_{z\to 0^+} \frac{1}{z^{2n+1}} \left(\frac{T_n'(1/z)}{T_n(1/z)} - \frac{nz}{\sqrt{1-z^2}}\right) &= \lim_{t\to+\infty} \frac{n\cosh^{2n+1}(t)(\tanh(nt)-1)}{\sinh(t)} \\
&= \frac{1}{2^{2n}} \lim_{t\to+\infty} \frac{n(e^t+e^{-t})^{2n+1}(-2e^{-nt})}{(e^t-e^{-t})(e^{nt}+e^{-nt})} \\
&= -\frac{n}{2^{2n-1}} \lim_{t\to+\infty} \frac{(1+e^{-2t})^{2n}}{(1-e^{-2t})(1+e^{-2nt})} \\
&= -\frac{n}{2^{2n-1}}
\end{aligned}$$

and the proof is complete. □