Problem 12168
(American Mathematical Monthly, Vol.127, March 2020)

Proposed by M. Lukarevski (North Macedonia).

Let a, b, and c be the side lengths of a triangle ABC with circumradius R and inradius r. Prove

$$\frac{2}{R} \leq \frac{\sec(A/2)}{a} + \frac{\sec(B/2)}{b} + \frac{\sec(C/2)}{c} \leq \frac{1}{r}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let s be the semiperimeter of the triangle and let F be its area. Then we have that

$$r = \frac{F}{s},\ R = \frac{abc}{4F},\ \sec(A/2) = \sqrt{\frac{bc}{s(s-a)}},\ \sec(B/2) = \sqrt{\frac{ca}{s(s-b)}},\ \sec(C/2) = \sqrt{\frac{ab}{s(s-c)}},$$

and the double inequalities can be written as

$$\frac{8F^2}{abc} \leq \frac{\sqrt{bc(s-b)(s-c)}}{a} + \frac{\sqrt{ca(s-c)(s-a)}}{b} + \frac{\sqrt{ab(s-a)(s-b)}}{c} \leq s.$$

We first show the inequality on the right: by AM-GM inequality

$$\frac{\sqrt{bc(s-b)(s-c)}}{a} \leq \frac{\frac{b+c}{2} \cdot \frac{2s-b-c}{2}}{a} = \frac{b+c}{4}$$

and therefore

$$\frac{\sqrt{bc(s-b)(s-c)}}{a} + \frac{\sqrt{ca(s-c)(s-a)}}{b} + \frac{\sqrt{ab(s-a)(s-b)}}{c} \leq \frac{b+c}{4} + \frac{c+a}{4} + \frac{a+b}{4} = s.$$

As regards the inequality on the left, let $x = \frac{1}{2}(b+c-a) > 0$, $y = \frac{1}{2}(c+a-b) > 0$, and $z = \frac{1}{2}(a+b-c) > 0$ then

$$a = x+y,\ b = y+z,\ c = z+x,\ s = x+y+z,\ F^2 = xyz(x+y+z)$$

and the inequality becomes

$$8xyz(x+y+z) \leq \text{RHS}$$

where

$$\text{RHS} := \sqrt{y(y+x)^3}\sqrt{z(z+x)^3} + \sqrt{x(x+y)^3}\sqrt{z(z+y)^3} + \sqrt{x(x+z)^3}\sqrt{y(y+z)^3}.$$

We have that

$$(2x+y)(x-y)^2 \Leftrightarrow \sqrt{y(y+x)^3} \geq \frac{y(y+3x)}{\sqrt{2}}$$

whence

$$\begin{aligned}\text{RHS} &\geq \frac{yz(y+3x)(z+3x)}{2} + \frac{xz(x+3y)(z+3y)}{2} + \frac{xy(x+3z)(y+3z)}{2}\\ &= \frac{x^2(y-z)^2 + y^2(x-z)^2 + z^2(x-y)^2}{4} + 8xyz(x+y+z)\\ &\geq 8xyz(x+y+z)\end{aligned}$$

and we are done. $\square$