Problem 12167
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Let S be the set of positive integers expressible as the sum of two nonzero Fibonacci numbers. Show that there are infinitely many six-term arithmetic progressions of numbers in S but only finitely many such seven-term arithmetic progressions.

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Solution. Note that $2 = F_2 + F_1 \in S$ and if $s \in S$ with $s > 2$ then there is a unique pair of integers i, j such that $s = F_i + F_j$ with $2 \le j < i$ where F_n denotes the n-th Fibonacci number. We call *type* of s the difference $i - j$. Therefore

$$S = \{2\} \cup \bigcup_{i=3}^{\infty} \{F_i + F_2, \ldots, F_i + F_{i-1}\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 21, \ldots\}.$$

An infinite family of 6-APs (arithmetic progressions) in S is:

$$\boxed{F_{n-3} + F_{n-4}, F_{n+1} + F_{n-2}, F_{n+2} + F_n, F_{n+3} + F_n, F_{n+3} + F_{n+2}, F_{n+4} + F_{n+1}}$$

where $n \ge 6$, the common difference is F_{n+1} and the sequence of types is $1, 3, 2, 3, 1, 3$.

The sequence $2, 3, 4, 5, 6, 7, 8$ is a 7-AP in S, but we will show that there are only finitely many such 7-APs in S.

Let $m = F_{n+h} + F_n$ be the midterm of a 3-AP in S, say l, m, r, then $l + r = 2m$. Note that $2m$ has a finite number of representations as a sum of 4 Fibonacci numbers and we are able to find them explicitely. It follows that we can list all the possible pairs $(l_1, r_1), \ldots, (l_k, r_k)$ systematically.

A 3-AP l_i, m, r_i in S can be extended to a 5-AP in S with m in the middle if and only if $l_i - (m - l_i), m, r_i + (r_i - m)$ is a 3-AP in S that is $l_i - (m - l_i) = l_j$ and $r_i + (r_i - m) = r_j$ for some j, which implies

$$2l_i = m + l_j$$

and therefore $2r_i = 2(m + m - l_i) = 4m - m - l_j = m + 2m - l_j = m + r_j$.

In order to check the equality $2l_i = m + l_j$ we use the Zeckendorf's theorem, i. e. every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers.

Now we are ready to classify all 5-APs in S with midterm $m = F_{n+h} + F_n$ such that $n > 8$ and $h \ge 1$ (and therefore $m \ge F_9 + F_8 = F_{10}$). We divide the analysis according to the value of the type h.

• $m = F_{n+h} + F_n$ with $h > 6$ then

$$\begin{aligned} 2m = 2F_{n+h} + 2F_n &= 2F_{n+h} + F_{n+1} + F_{n-2} = F_{n+h+1} + F_{n+h-2} + 2F_n \\ &= F_{n+h+1} + F_{n+h-2} + F_{n+1} + F_{n-2}. \end{aligned}$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n+1} + F_{n-2}$	$F_{n+h+1} + F_{n+h-2}$	$F_{n+2} + F_n + F_{n-2}$	$F_{n+h} + F_{n+2} + F_{n-2}$
$F_{n+h-2} + F_{n-2}$	$F_{n+h+1} + F_{n+1}$	$F_{n+h-1} + F_{n+h-3} + F_{n-1} + F_{n-4}$	$F_{n+h-2} + F_{n+h} + F_n + F_{n-2}$
$F_{n+h-2} + F_n$	$F_{n+h+1} + F_n$	$F_{n+h-1} + F_{n+h-4} + F_{n+1} + F_{n-2}$	$F_{n+h} + F_{n+h-2} + F_{n+1} + F_{n-2}$
$F_{n+h-2} + F_{n+1}$	$F_{n+h+1} + F_{n-2}$	$F_{n+h-1} + F_{n+h-4} + F_{n+2} + F_{n-1}$	$F_{n+h} + F_{n+h-2} + F_{n+2}$
$F_{n+h} + F_{n-2}$	$F_{n+h+1} + F_{n+1}$	$F_{n+h+1} + F_{n+h-2} + F_{n-1} + F_{n-4}$	$F_{n+h+1} + F_{n+h-2} + F_n + F_{n-2}$

The equation $2l_i = m + l_j$ is never satisfied.

• If $m = F_{n+6} + F_n$ then

$$\begin{aligned}2m &= 2F_{n+6} + 2F_n = 2F_{n+6} + F_{n+1} + F_{n-2} = F_{n+7} + F_{n+4} + 2F_n \\ &= F_{n+7} + F_{n+4} + F_{n+1} + F_{n-2} = F_{n+7} + 2F_{n+3} + F_{n-2}.\end{aligned}$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n+1} + F_{n-2}$	$F_{n+7} + F_{n+4}$	$F_{n+2} + F_n + F_{n-2}$	$F_{n+6} + F_{n+2} + F_{n-2}$
$F_{n+3} + F_{n-2}$	$F_{n+7} + F_{n+3}$	$F_{n+4} + F_{n+1} + F_{n-1} + F_{n-4}$	$F_{n+6} + F_{n+3} + F_n + F_{n-2}$
$F_{n+4} + F_{n-2}$	$F_{n+7} + F_{n+1}$	$F_{n+5} + F_{n+2} + F_{n-1} + F_{n-4}$	$F_{n+6} + F_{n+4} + F_n + F_{n-2}$
$F_{n+4} + F_n$	$F_{n+7} + F_n$	$F_{n+5} + F_{n+3} + F_{n-2}$	$F_{n+6} + F_{n+4} + F_{n+1} + F_{n-2}$
$F_{n+4} + F_{n+1}$	$F_{n+7} + F_{n-2}$	$F_{n+5} + F_{n+3} + F_{n+1}$	$F_{n+6} + F_{n+4} + F_{n+2}$
$F_{n+6} + F_{n-2}$	$F_{n+6} + F_{n+1}$	$F_{n+7} + F_{n+4} + F_{n-1} + F_{n-4}$	$F_{n+7} + F_{n+4} + F_n + F_{n-2}$

Therefore the equation $2l_i = m + l_j$ is never satisfied.

• If $m = F_{n+5} + F_n$ then

$$\begin{aligned}2m &= 2F_{n+5} + 2F_n = 2F_{n+5} + F_{n+1} + F_{n-2} = F_{n+6} + F_{n+3} + 2F_n \\ &= F_{n+6} + F_{n+3} + F_{n+1} + F_{n-2} = F_{n+6} + 2F_{n+2} + F_n.\end{aligned}$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n+1} + F_{n-2}$	$F_{n+6} + F_{n+3}$	$F_{n+2} + F_n + F_{n-2}$	$F_{n+5} + F_{n+2} + F_{n-2}$
$F_{n+2} + F_n$	$F_{n+6} + F_{n+2}$	$F_{n+4} + F_{n-2}$	$F_{n+5} + F_{n+3} + F_{n-2}$
$F_{n+3} + F_{n-2}$	$F_{n+6} + F_{n+1}$	$F_{n+4} + F_{n+1} + F_{n-1} + F_{n-4}$	$F_{n+5} + F_{n+3} + F_n + F_{n-2}$
$F_{n+3} + F_n$	$F_{n+6} + F_n$	$F_{n+4} + F_{n+2} + F_n$	$F_{n+5} + F_{n+3} + F_{n+1} + F_{n-2}$
$F_{n+3} + F_{n+1}$	$F_{n+6} + F_{n-2}$	$F_{n+5} + F_{n-1}$	F_{n+6}
$F_{n+5} + F_{n-2}$	$F_{n+5} + F_{n+1}$	$F_{n+6} + F_{n+3} + F_{n-1} + F_{n-4}$	$F_{n+6} + F_{n+3} + F_n + F_{n-2}$

Therefore the equation $2l_i = m + l_j$ is never satisfied.

• If $m = F_{n+4} + F_n$ then

$$\begin{aligned}2m &= 2F_{n+4} + 2F_n = 2F_{n+4} + F_{n+1} + F_{n-2} = F_{n+5} + F_{n+2} + 2F_n \\ &= F_{n+5} + F_{n+2} + F_{n+1} + F_{n-2} = F_{n+4} + 2F_{n+3} + F_{n-2} \\ &= F_{n+5} + F_{n+3} + F_{n-3} + F_{n-4}.\end{aligned}$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n-3} + F_{n-4}$	$F_{n+5} + F_{n+3}$	$F_{n-1} + F_{n-4}$	$F_{n+4} + F_n + F_{n-2}$
$F_{n+1} + F_{n-2}$	$F_{n+5} + F_{n+2}$	$F_{n+2} + F_n + F_{n-2}$	$F_{n+4} + F_{n+2} + F_{n-2}$
$F_{n+2} + F_{n-2}$	$F_{n+5} + F_{n+1}$	$F_{n+3} + F_{n+1} + F_{n-4}$	$F_{n+4} + F_{n+2} + F_n + F_{n-2}$
$F_{n+2} + F_n$	$F_{n+5} + F_n$	$F_{n+4} + F_{n-2}$	$F_{n+5} + F_{n-2}$
$F_{n+2} + F_{n+1}$	$F_{n+5} + F_{n-2}$	$F_{n+4} + F_{n+1}$	$F_{n+5} + F_n$
$F_{n+3} + F_{n-4}$	$F_{n+5} + F_{n-3}$	$F_{n+4} + F_{n+1} + F_{n-3} + F_{n-6}$	$F_{n+5} + F_n + F_{n-4}$
$F_{n+3} + F_{n-3}$	$F_{n+5} + F_{n-4}$	$F_{n+4} + F_{n+1} + F_{n-2} + F_{n-5}$	$F_{n+5} + F_n + F_{n-3}$
$F_{n+3} + F_{n-2}$	$F_{n+4} + F_{n+3}$	$F_{n+4} + F_{n+1} + F_{n-1} + F_{n-4}$	$F_{n+5} + F_n + F_{n-2}$
$F_{n+4} + F_{n-2}$	$F_{n+4} + F_{n+1}$	$F_{n+5} + F_{n+2} + F_{n-1} + F_{n-4}$	$F_{n+5} + F_{n+2} + F_n + F_{n-2}$

Therefore the equation $2l_i = m + l_j$ is never satisfied.

• If $m = F_{n+3} + F_n$ then

$$2m = 2F_{n+3} + 2F_n = 4F_{n+2} = 2F_{n+3} + F_{n+1} + F_{n-2} = F_{n+4} + F_{n+1} + 2F_n$$
$$= F_{n+4} + 2F_{n+1} + F_{n-2} = F_{n+4} + F_{n+2} + 2F_{n-2}$$
$$= F_{n+4} + F_{n+2} + F_{n-1} + F_{n-4}.$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n-1} + F_{n-2}$	$F_{n+4} + F_{n+2}$	$F_{n+1} + F_{n-2}$	$F_{n+3} + F_{n+1} + F_{n-2}$
$F_{n+1} + F_{n-2}$	$F_{n+4} + F_{n+1}$	$F_{n+2} + F_n + F_{n-2}$	$\mathbf{F_{n+4} + F_{n-2}}$
$F_{n+1} + F_n$	$F_{n+4} + F_n$	$F_{n+3} + F_n$	$F_{n+4} + F_n$
$F_{n+2} + F_{n-2}$	$F_{n+4} + F_{n-1}$	$F_{n+3} + F_{n+1} + F_{n-4}$	$F_{n+4} + F_n + F_{n-2}$
$F_{n+2} + F_{n-1}$	$F_{n+4} + F_{n-2}$	$F_{n+3} + F_{n+1} + F_{n-1}$	$F_{n+4} + F_{n+1}$
$F_{n+2} + F_n$	$F_{n+3} + F_{n+2}$	$\mathbf{F_{n+4} + F_{n-2}}$	$F_{n+4} + F_{n+1} + F_{n-2}$
$F_{n+3} + F_{n-2}$	$F_{n+3} + F_{n+1}$	$F_{n+4} + F_{n+1} + F_{n-1} + F_{n-4}$	$F_{n+4} + F_{n+2} + F_{n-2}$

The equation $2l_i = m + l_j$ is satisfied for $i = 6$ and $j = 2$ and we have a 5-AP:

$$\boxed{F_{n+1} + F_{n-2}, F_{n+2} + F_n, F_{n+3} + F_n, F_{n+3} + F_{n+2}, F_{n+4} + F_{n+1}}$$

where the common difference is F_{n+1} and the sequence of types is $3, 2, 3, 1, 3$.

• If $m = F_{n+2} + F_n$ then

$$2m = 2F_{n+2} + 2F_n = 2F_{n+2} + F_{n+1} + F_{n-2} = F_{n+3} + 3F_n$$
$$= F_{n+3} + F_{n+1} + F_n + F_{n-2} = F_{n+3} + F_{n+2} + F_{n-3} + F_{n-4}$$
$$= F_{n+4} + 2F_{n-4} + F_{n-5} = F_{n+4} + F_{n-3} + F_{n-5} + F_{n-6}.$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n-5} + F_{n-6}$	$F_{n+4} + F_{n-3}$	$F_{n-3} + F_{n-6}$	$F_{n+2} + F_n + F_{n-4}$
$F_{n-4} + F_{n-5}$	$F_{n+4} + F_{n-4}$	$F_{n-2} + F_{n-5}$	$F_{n+2} + F_n + F_{n-3}$
$F_{n-3} + F_{n-6}$	$F_{n+4} + F_{n-5}$	$F_{n-2} + F_{n-4} + F_{n-6}$	$F_{n+2} + F_n + F_{n-3} + F_{n-6}$
$F_{n-3} + F_{n-4}$	$F_{n+3} + F_{n+2}$	$F_{n-1} + F_{n-4}$	$\mathbf{F_{n+2} + F_n + F_{n-2}}$
$F_{n-3} + F_{n-5}$	$F_{n+4} + F_{n-6}$	$F_{n-1} + F_{n-7}$	$F_{n+2} + F_n + F_{n-3} + F_{n-5}$
$F_n + F_{n-2}$	$F_{n+3} + F_{n+1}$	$F_{n+2} + F_{n-4}$	$F_{n+3} + F_{n-1} + F_{n-4}$
$F_{n+1} + F_{n-2}$	$F_{n+3} + F_n$	$\mathbf{F_{n+2} + F_n + F_{n-2}}$	$F_{n+3} + F_n + F_{n-2}$
$F_{n+1} + F_n$	$F_{n+3} + F_{n-2}$	$F_{n+3} + F_n$	$F_{n+3} + F_{n+1} + F_{n-2}$
$F_{n+2} + F_{n-4}$	$F_{n+3} + F_{n-3}$	$F_{n+3} + F_n + F_{n-3} + F_{n-6}$	$F_{n+3} + F_{n+1} + F_{n-2} + F_{n-4}$
$F_{n+2} + F_{n-3}$	$F_{n+3} + F_{n-4}$	$F_{n+3} + F_n + F_{n-2} + F_{n-5}$	$F_{n+3} + F_{n+1} + F_{n-1}$
$F_{n+2} + F_{n-2}$	$F_{n+2} + F_{n+1}$	$F_{n+3} + F_{n+1} + F_{n-4}$	$F_{n+3} + F_{n+1} + F_{n-1} + F_{n-4}$

The equation $2l_i = m + l_j$ is satisfied for $i = 7$ and $j = 4$ and we have a 5-AP:

$$\boxed{F_{n-3} + F_{n-4}, F_{n+1} + F_{n-2}, F_{n+2} + F_n, F_{n+3} + F_n, F_{n+3} + F_{n+2}}$$

where the common difference is F_{n+1} and the sequence of types is $1, 3, 2, 3, 1$.

• If $m = F_{n+1} + F_n$ then

$$\begin{aligned}2m &= 2F_{n+1} + 2F_n = 3F_{n+1} + F_{n-2} = F_{n+2} + 2F_n + F_{n-1} \\ &= F_{n+2} + F_{n+1} + F_{n-1} + F_{n-2} = F_{n+3} + 2F_{n-2} + F_{n-3} \\ &= F_{n+3} + F_{n-1} + F_{n-3} + F_{n-4}.\end{aligned}$$

l_i	r_i	$2l_i$	$m + l_i$
$F_{n-3} + F_{n-4}$	$F_{n+3} + F_{n-1}$	$F_{n-1} + F_{n-4}$	$F_{n+2} + F_{n-2}$
$F_{n-2} + F_{n-3}$	$F_{n+3} + F_{n-2}$	$F_n + F_{n-3}$	$\mathbf{F_{n+2} + F_{n-1}}$
$F_{n-1} + F_{n-4}$	$F_{n+3} + F_{n-3}$	$F_n + F_{n-2} + F_{n-4}$	$F_{n+2} + F_{n-1} + F_{n-4}$
$F_{n-1} + F_{n-3}$	$F_{n+3} + F_{n-4}$	$F_{n+1} + F_{n-5}$	$F_{n+2} + F_{n-1} + F_{n-3}$
$F_{n-1} + F_{n-2}$	$F_{n+2} + F_{n+1}$	$F_{n+1} + F_{n-2}$	$F_{n+2} + F_n$
$F_n + F_{n-1}$	$F_{n+2} + F_n$	$\mathbf{F_{n+2} + F_{n-1}}$	F_{n+3}
$F_{n+1} + F_{n-2}$	$F_{n+2} + F_{n-1}$	$F_{n+2} + F_n + F_{n-2}$	$F_{n+3} + F_{n-2}$
$F_{n+1} + F_{n-1}$	$F_{n+2} + F_{n-2}$	$F_{n+3} + F_{n-3}$	$F_{n+3} + F_{n-1}$

The equation $2l_i = m + l_j$ is satisfied for $i = 6$ and $j = 2$ and we have a 5-AP:

$$\boxed{F_{n-2} + F_{n-3}, F_n + F_{n-1}, F_{n+1} + F_n, F_{n+2} + F_n, F_{n+3} + F_{n-2}}$$

where the common difference is F_n and the sequence of types is $1, 1, 1, 2, 1$.

For the sake of clarity, we summarize our results as follows if $m = F_{n+h} + F_n \geq F_{10}$ is the midterm of a 5-AP then we have just three possible sequence of types:

$$3, 2, \mathbf{3}, 1, 3, \quad 1, 3, \mathbf{2}, 3, 1, \quad 1, 1, \mathbf{1}, 2, 1.$$

Finally, assume by contradiction that $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ is a 7-AP in S with $s_3 \geq F_{10}$. Then s_3, s_4, and s_5 are the midterms of three 5-APs and, by using the above results, it follows that:

- If the type of the midterm s_4 is 1 then the type of s_5 is 2, but then, by considering s_5 as midterm, the type of s_4 should be 3. Contradiction.
- If the type of the midterm s_4 is 2 then the type of s_3 is 3, but then, by considering s_3 as midterm, the type of s_4 should be 1. Contradiction.
- If the type of the midterm s_4 is 3 then the type of s_5 is 1, but then, by considering s_5 as midterm, the type of s_4 should be 1. Contradiction.

Therefore such 7-AP does not exist and the proof is complete. □

Remark. Among the 5-APs that we found before, only two can be extended to a 6-AP: when $h = 2$ the 5-AP can be extended on the right and when $h = 3$ the 5-AP can be extended on the left. In both cases we find

$$\boxed{F_{n-3} + F_{n-4}, F_{n+1} + F_{n-2}, F_{n+2} + F_n, F_{n+3} + F_n, F_{n+3} + F_{n+2}, F_{n+4} + F_{n+1}}$$

which has been mentioned before.