Problem 12165
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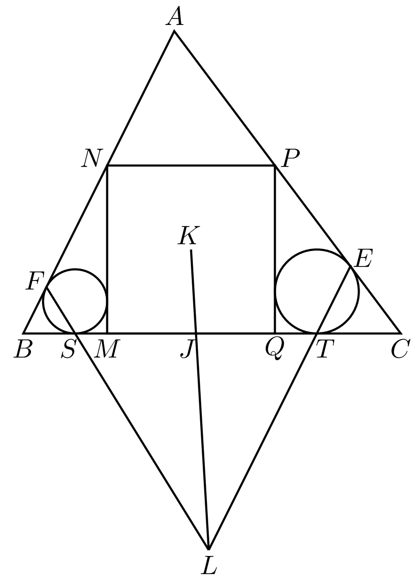
Proposed by Tran Quang Hung and Nguyen Minh Ha (Vietnam).

Let $MNPQ$ be a square with center K inscribed in triangle ABC with N and P lying on sides AB and AC, respectively, while M and Q lie on side BC. Let the incircle of BMN touch side BM at S and side BN at F, and let the incircle of CQP touch side CQ at T and side CP at E. Let L be the point of intersection of lines FS and ET. Prove that KL bisects the segment ST.



Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We set up a coordinate system with the x-axis along the side BC such that $A = (0, h)$. Let $B = (-a_1, 0)$ and $C = (a_2, 0)$ with $a_1 + a_2 = a = |BC|$, $b = |AC|$ and $c = |AB|$ and let l be the side of the square. Since the triangle ABC is similar to the triangle ANP we have that $\frac{h}{a} = \frac{h-l}{l}$ and we find that $l = \frac{ah}{a+h}$. Then the four vertices of the square and its center are

$$M = \frac{l}{a}(-a_1, 0)\,,\ Q = \frac{l}{a}(a_2, 0)\,,\ N = \frac{l}{a}(-a_1, a)\,,\ P = \frac{l}{a}(a_2, a)\,,\ K = \frac{l}{2a}(a_2 - a_1, a)\,.$$

Next we find the radii of the the incircles of the right triangles BMN and CQP:

$$r_1 = \frac{la_1}{a_1 + h + c} \quad , \quad r_2 = \frac{la_2}{a_2 + h + b}.$$

Therefore $S = (x_M - r_1, 0)$ and $T = (x_Q + r_2, 0)$ which implies

$$J = \frac{S + T}{2} = \frac{l}{2a}\left(a_2 - a_1 + \frac{a_2 a}{a_2 + h + b} - \frac{a_1 a}{a_1 + h + c}, 0\right).$$

Moreover the lines FS and ET are respectively

$$y = -\frac{x - x_S}{\tan\left(\frac{\angle B}{2}\right)} = -\frac{(a_1 + c)(x - x_S)}{h}, \quad y = \frac{x - x_T}{\tan\left(\frac{\angle C}{2}\right)} = \frac{(a_2 + b)(x - x_T)}{h}$$

and their intersection point is

$$L = -\frac{l}{2a}\left(a_1 - a_2 + 2c - 2b, \frac{2(a_2 + b)(a_1 + c)}{h} - 2h - a\right).$$

Finally, it is easy to verify that $(1 - t)K + tL = J$ for

$$t = \frac{ah}{2((a_2 + b)(a_1 + c) - h^2)}$$

and we are done. $\square$