Problem 12163
(American Mathematical Monthly, Vol.127, February 2020)

Proposed by T. Speckhofer (Austria).

Let $\mathbb{R}^n$ have the usual dot product and norm. When $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\Sigma v = x_1 + \cdots + x_n$. Prove

$$\|v\|^2\|w\|^2 \geq (v \cdot w)^2 + \frac{(\|v\| \, |\Sigma w| - \|w\| \, |\Sigma v|)^2}{n}$$

for all $v, w \in \mathbb{R}^n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show the more general inequality

$$\left(\|v\|^2\|w\|^2 - (v \cdot w)^2\right) \|u\|^2 \geq \|(w, u)v - (v, u)w\|^2$$

where $u \in \mathbb{R}^n$.

By letting $u = (1, \ldots, 1)$, we have $\|u\|^2 = n$, $(v, u) = \sum v$, and $(w, u) = \sum w$ and we find

$$(\|v\|^2\|w\|^2 - (v \cdot w)^2)\, n \geq \|(\Sigma w)v - (\Sigma v)w\|^2 \geq (|\Sigma w| \, \|v\| - |\Sigma v| \, \|w\|)^2$$

which is equivalent to the given inequality.

If v and w are linearly dependent then $\|v\|^2\|w\|^2 = (v \cdot w)^2$ the inequality holds. We assume now that v and w are linearly independent. Then

$$u = \alpha v + \beta w + z$$

where $z \perp v$, $z \perp w$ and $\alpha, \beta \in \mathbb{R}$. Moreover

$$\begin{cases} (v, u) = \alpha\|v\|^2 + \beta(v, w) \\ (w, u) = \alpha(v, w) + \beta\|w\|^2 \end{cases} .$$

and by solving the linear system we find

$$\alpha = \frac{(v, u)\|w\|^2 - (w, u)(v, w)}{\|v\|^2\|w\|^2 - (v, w)^2} \quad \text{and} \quad \beta = \frac{(w, u)\|v\|^2 - (v, u)(v, w)}{\|v\|^2\|w\|^2 - (v, w)^2}.$$

Hence

$$\begin{aligned} \left(\|v\|^2\|w\|^2 - (v \cdot w)^2\right) \|u\|^2 &= \left(\|v\|^2\|w\|^2 - (v \cdot w)^2\right) (\|\alpha v + \beta w\|^2 + \|z\|^2) \\ &\geq \left(\|v\|^2\|w\|^2 - (v \cdot w)^2\right) (\|\alpha v + \beta w\|^2) \\ &= \left(\|v\|^2\|w\|^2 - (v \cdot w)^2\right) (\alpha^2\|v\|^2 + \beta^2\|w\|^2 + 2\alpha\beta(v, w)) \\ &= (w, u)^2\|v\|^2 + (v, u)^2\|w\|^2 - 2(w, u)(v, u)(v, w) \\ &= \|(w, u)v - (v, u)w\|^2. \end{aligned}$$

□