Problem 12162
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Proposed by Dao Thanh Oai (Vietnam) and L. Giugiuc (Romania).

Consider a triangle with sides of lengths a, b, and c and with area S. Prove

$$\sqrt{a^2+b^2-4S}+\sqrt{a^2+c^2-4S} \geq \sqrt{b^2+c^2-4S}$$

and determine when equality holds.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $4S = 2ab\sin(C) \leq 2ab$, it follows that $a^2+b^2-4S \geq 0$. Similarly for the other square root arguments. After squaring both sides, the given inequality is equivalent to

$$a^2+b^2-4S+a^2+c^2-4S+2\sqrt{a^2+b^2-4S}\sqrt{a^2+c^2-4S} \geq b^2+c^2-4S$$

that is

$$\sqrt{a^2+b^2-4S}\sqrt{a^2+c^2-4S} \geq 2S-a^2.$$

The inequality holds strictly when $2S-a^2<0$, otherwise by squaring again we find

$$(a^2+b^2-4S)(a^2+c^2-4S) \geq (2S-a^2)^2$$

that is

$$12S^2+a^2b^2+b^2c^2+c^2a^2 \geq 4S(a^2+b^2+c^2).$$

By squaring one more time, it remains to show that

$$(12S^2+a^2b^2+b^2c^2+c^2a^2)^2 \geq 16S^2(a^2+b^2+c^2)$$

which holds because, by Heron's formula,

$$16S^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$$

and it follows that

$$(12S^2+a^2b^2+b^2c^2+c^2a^2)^2-16S^2(a^2+b^2+c^2) = \frac{1}{4}\left(a^4+b^4+c^4-24S^2\right)^2 \geq 0.$$

Equality holds if the given side lengths a, b, c satisfy $a^4+b^4+c^4=24S^2$ with $2S \geq a^2$. $\square$