Problem 12161
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Proposed by J. H. Santiago (Mexico).

Let $N(C)$ be the number of pairs $(u, v) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $u^2 + uv + v^2 = C$. Prove that 6 divides $N(C)$ for every positive integer C.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$S_C = \{(u, v) \in \mathbb{Z} \times \mathbb{Z} \ : \ u^2 + uv + v^2 = C\}.$$

Then $N(C)$, i.e. the cardinality of S_C, is finite for any positive integer C because

$$C = u^2 + uv + v^2 \geq |uv| \implies u, v \in [-C, C] \cap \mathbb{Z} \implies N(C) \leq (2C + 1)^2.$$

Morever the trasformation $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $T(u, v) = (u + v, -u)$ is a bijection such that $T(S_C) = S_C$:

$$(u + v)^2 + (u + v)(-u) + (-u)^2 = C \Leftrightarrow u^2 + uv + v^2 = C.$$

It follows that given $(u, v) \in S_C$ then

$$\begin{aligned} T(u, v) &= (u + v, -u), & T^2(u, v) &= (v, -u - v), & T^3(u, v) &= (-u, -v), \\ T^4(u, v) &= (-u - v, u), & T^5(u, v) &= (-v, u + v), & T^6(u, v) &= (u, v), \end{aligned}$$

are 6 distinct elements of S_C and therefore 6 divides $N(C)$. □

Remark. If C is a positive integer then it can be shown that $N(C) > 0$ if and only if C has the form

$$C := 3^m \prod_{i=1}^{r} p_i^{\alpha_i} \prod_{j=1}^{s} q_j^{2\beta_j}$$

where $m, r, s, \alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_s$ are non-negative integers, $p_1 < p_2 < \ldots < p_r$ are primes congruent to 1 modulo 3, and $q_1 < q_2 < \ldots < q_s$ are primes congruent to 2 modulo 3 and

$$N(C) := 6 \prod_{i=1}^{r} (\alpha_i + 1).$$

The first values of $C > 0$ with $N(C) > 0$ are the so-called *Loeschian numbers* (see OEIS sequence A003136:

$$1(6), 3(6), 4(6), 7(12), 9(6), 12(6), 13(12), 16(6), 19(12), 21(12), 25(6), 27(6), 28(12), \ldots$$

where the values in parentheses are the corresponding $N(C)$.