Problem 12160
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Let F_n be the nth Fibonacci number, and let L_n be the nth Lucas number. Prove

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}=5^n \quad \textit{and} \quad \sum_{k=0}^{n}\binom{2n+1}{n-k}L_{2k+1}=\sum_{k=0}^{n}\binom{2k}{k}5^{n-k}$$

for all $n\in\mathbb{N}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$\begin{aligned} f(x,y) &:= \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{2n+1}{n-k}y^{2k+1}=\sum_{k=0}^{\infty}y^{2k+1}\sum_{n=k}^{\infty}\binom{2n+1}{n-k}x^n \\ &= y\sum_{k=0}^{\infty}(xy^2)^k\sum_{n=0}^{\infty}\binom{2n+(2k+1)}{n}x^n \\ &= \frac{2y}{\sqrt{1-4x}(1+\sqrt{1-4x})}\sum_{k=0}^{\infty}\left(\frac{4xy^2}{(1+\sqrt{1-4x})^2}\right)^k \\ &= \frac{2y}{\sqrt{1-4x}(1+\sqrt{1-4x})\left(1-\frac{4xy^2}{(1+\sqrt{1-4x})^2}\right)} \\ &= \frac{y(1+\sqrt{1-4x})}{\sqrt{1-4x}\left(1+\sqrt{1-4x}-2x(1+y^2)\right)} \end{aligned}$$

where we used the known identity

$$\sum_{n=0}^{\infty}\binom{2n+m}{n}x^n=\frac{1}{\sqrt{1-4x}}\left(\frac{2}{1+\sqrt{1-4x}}\right)^m$$

(see (5.72) in *Concrete Mathematics* by D. E. Knuth, O. Patashnik, and R. Graham).
Since

$$F_{2k+1}=\frac{a^{2k+1}-b^{2k+1}}{\sqrt{5}} \quad \text{and} \quad L_{2k+1}=a^{2k+1}+b^{2k+1}$$

with $a:=(1+\sqrt{5})/2$ and $b:=(1-\sqrt{5})/2$, it follows that

$$\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}=\frac{f(x,a)-f(x,b)}{\sqrt{5}},$$

and

$$\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{2n+1}{n-k}L_{2k+1}=f(x,a)+f(x,b).$$

On the other hand

$$g(x):=\sum_{n=0}^{\infty}5^nx^n=\frac{1}{1-5x}.$$

and

$$h(x):=\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{2k}{k}5^{n-k}=\frac{1}{\sqrt{1-4x}\,(1-5x)}.$$

Hence, in order to show our identities, it remains to check that

$$f(x,a)-f(x,b)=\sqrt{5}\,g(x) \quad \text{and} \quad f(x,a)+f(x,b)=h(x)$$

which can be easily verified. □