**Problem 12158**
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Proposed by H. Grandmontagne (France).

*Prove*

$$\int_0^1 \frac{(\ln(x))^2 \arctan(x)}{1+x}\,dx = \frac{21\pi\zeta(3)}{64} - \frac{\pi^2 G}{24} - \frac{\pi^3\ln(2)}{32}$$

*where* $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ *is Catalan's constant.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* By integration by parts, we have that

$$\begin{aligned}\int_0^1 \frac{(\ln(x))^2 \arctan(x)}{1+x}\,dx &= \left[(\ln(x))^2 \arctan(x)\ln(1+x)\right]_0^1 - \int_0^1 D\left((\ln(x))^2\arctan(x)\right)\ln(1+x)\,dx\\ &= 0 - 2I_1 - I_2\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_0^1 \frac{\arctan(x)\ln(x)\ln(1+x)}{x}\,dx\\ &= [\arctan(x)(-\ln(x)\operatorname{Li}_2(-x) + \operatorname{Li}_3(-x)]_0^1 + \int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x)}{1+x^2}\,dx - \int_0^1 \frac{\operatorname{Li}_3(-x)}{1+x^2}\,dx\\ &= -\frac{3\pi\zeta(3)}{16} + \int_0^1 \frac{\ln(x)\operatorname{Li}_2(-x)}{1+x^2}\,dx - \int_0^1 \frac{\operatorname{Li}_3(-x)}{1+x^2}\,dx\end{aligned}$$

(recall that $\operatorname{Li}_1(x) = -\ln(1-x)$ and $D(\operatorname{Li}_{s+1}(x)) = \frac{1}{x}\operatorname{Li}_s(x)$), and

$$\begin{aligned}I_2 &= \int_0^1 \frac{(\ln(x))^2\ln(1+x)}{1+x^2}\,dx = F(1) - F(0) = \int_0^1 F'(a)\,da\\ &= \frac{\pi^3}{16}\int_0^1 \frac{a}{1+a^2}\,da + \frac{3\zeta(3)}{16}\int_0^1 \frac{1}{1+a^2}\,da + 2\int_0^1 \frac{\operatorname{Li}_3(-a)}{1+a^2}\,da\\ &= \frac{\pi^3\ln(2)}{32} + \frac{3\pi\zeta(3)}{64} + 2\int_0^1 \frac{\operatorname{Li}_3(-a)}{1+a^2}\,da\end{aligned}$$

with

$$F(a) = \int_0^1 \frac{(\ln(x))^2\ln(1+ax)}{1+x^2}\,dx$$

and

$$\begin{aligned}F'(a) &= \int_0^1 \frac{x(\ln(x))^2}{(1+x^2)(1+ax)}\,dx = \int_0^1 \frac{(\ln(x))^2}{1+a^2}\left(\frac{a}{1+x^2} + \frac{x}{1+x^2} - \frac{a}{1+ax}\right)dx\\ &= \frac{a}{1+a^2}\int_0^1 \frac{(\ln(x))^2}{1+x^2}\,dx + \frac{1}{1+a^2}\int_0^1 \frac{x(\ln(x))^2}{1+x^2}\,dx - \frac{a}{1+a^2}\int_0^1 \frac{(\ln(x))^2}{1+ax}\,dx\\ &= \frac{a}{1+a^2}\cdot\frac{\pi^3}{16} + \frac{1}{1+a^2}\cdot\frac{3\zeta(3)}{16} + \frac{2\operatorname{Li}_3(-a)}{1+a^2}.\end{aligned}$$

Note that

$$\int_0^1 \frac{(\ln(x)^2}{1+x^2}\,dx = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}(\ln x)^2\,dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{16},$$

$$\int_0^1 \frac{x(\ln(x)^2}{1+x^2}\,dx = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}(\ln x)^2\,dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^3} = \frac{3\zeta(3)}{16},$$

and

$$a\int_0^1 \frac{(\ln(x)^2}{1+ax}\,dx = -\sum_{n=0}^{\infty}(-a)^{n+1}\int_0^1 x^n(\ln x)^2\,dx = -2\sum_{n=0}^{\infty}\frac{(-a)^{n+1}}{(n+1)^3} = -2\,\mathrm{Li}_3(-a).$$

Hence, we finally find

$$\begin{aligned}\int_0^1 \frac{(\ln(x))^2\arctan(x)}{1+x}\,dx &= -2I_1 - I_2\\ &= \frac{3\pi\zeta(3)}{8} - 2\int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(-x)}{1+x^2}\,dx + 2\int_0^1 \frac{\mathrm{Li}_3(-x)}{1+x^2}\,dx\\ &\quad - \frac{\pi^3\ln(2)}{32} - \frac{3\pi\zeta(3)}{64} - 2\int_0^1 \frac{\mathrm{Li}_3(-a)}{1+a^2}\,da\\ &= \frac{21\pi\zeta(3)}{64} - \frac{\pi^2 G}{24} - \frac{\pi^3\ln(2)}{32}\end{aligned}$$

where

$$\begin{aligned}\int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(-x)}{1+x^2}\,dx &= \int_0^1 \ln(x)\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\sum_{k=0}^{\infty}(-1)^k x^{2k}\,dx\\ &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+n+1)^2}\\ &= -\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+2n+1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+2n)^2}\\ &= -\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=n}^{\infty}\frac{(-1)^k}{(2k+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sum_{k=n}^{\infty}\frac{(-1)^k}{k^2}\\ &= -\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=n}^{\infty}\frac{(-1)^k}{(2k+1)^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=0}^{k-1}\frac{(-1)^n}{(2n+1)^2}\\ &= \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = \frac{1}{4}\cdot\frac{\pi^2}{12}\cdot G = \frac{\pi^2 G}{48}.\end{aligned}$$

□