Problem 12156
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Proposed by H. Ohtsuka (Japan).

For positive integers m and n and nonnegative integers r and s, prove

$$\sum_{0\le j_1\le\cdots\le j_m\le r}\frac{\binom{n+s}{n}\binom{n+j_1}{n}\binom{s+j_1}{s}}{\prod_{i=1}^m(n+j_i)}=\sum_{0\le j_1\le\cdots\le j_m\le s}\frac{\binom{n+r}{n}\binom{n+j_1}{n}\binom{r+j_1}{r}}{\prod_{i=1}^m(n+j_i)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first show, by induction with respect to m, that

$$\sum_{0\le j_1\le\cdots\le j_m\le s}\frac{\binom{n+j_1}{n}\binom{j_1}{k}}{\prod_{i=1}^m(n+j_i)}=\frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m}\tag{1}$$

where k is a non-negative integer. Let $F_s(m,n,k)$ be the left-hand side.
Base case: if $m=1$ then

$$F_s(1,n,k)=\sum_{0\le j_1\le s}\frac{\binom{n+j_1}{n}\binom{j_1}{k}}{(n+j_1)}=\sum_{0\le j_1\le s}\frac{\binom{n+j_1}{n}\binom{j_1}{k}-\binom{n+j_1-1}{n}\binom{j_1-1}{k}}{n+k}=\frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)}.$$

Inductive step: for $m>1$,

$$\begin{aligned}F_s(m,n,k)&=\sum_{t=0}^{s}\left(F_t(m,n,k)-F_{t-1}(m,n,k)\right)=\sum_{t=0}^{s}\frac{F_t(m-1,n,k)}{n+t}\\&=\sum_{t=0}^{s}\frac{\frac{\binom{n+t}{n}\binom{t}{k}}{(n+k)^{m-1}}}{n+t}=\frac{F_s(1,n,k)}{(n+k)^{m-1}}=\frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m}.\end{aligned}$$

Let $R(m,n,r,s)$ be the right-hand side of the given identity, then we have to prove that

$$R(m,n,s,r)=R(m,n,r,s).$$

Since by Vandermonde's identity,

$$\binom{r+j_1}{r}=\sum_{k=0}^{r}\binom{r}{r-k}\binom{j_1}{k}=\sum_{k=0}^{r}\binom{r}{k}\binom{j_1}{k},$$

by using (1), we find

$$\begin{aligned}R(m,n,r,s)&=\sum_{0\le j_1\le\cdots\le j_m\le s}\frac{\binom{n+r}{n}\binom{n+j_1}{n}\sum_{k=0}^{r}\binom{r}{k}\binom{j_1}{i}}{\prod_{i=1}^m(n+j_i)}\\&=\binom{n+r}{n}\sum_{k=0}^{r}\binom{r}{k}\sum_{0\le j_1\le\cdots\le j_m\le s}\frac{\binom{n+j_1}{n}\binom{j_1}{k}}{\prod_{i=1}^m(n+j_i)}\\&=\binom{n+r}{n}\sum_{k=0}^{r}\binom{r}{k}\cdot\frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m}\\&=\binom{n+r}{n}\binom{n+s}{n}\sum_{k=0}^{\min(r,s)}\frac{\binom{r}{k}\binom{s}{k}}{(n+k)^m}\end{aligned}$$

which is symmetric with respect to r and s and we are done. □