Problem 12154
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Proposed by M. Lukarevski (North Macedonia).

Let r_a, r_b, and r_c be the exradii of a triangle with circumradius R and inradius r. Prove

$$\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}\geq 2-\frac{r}{R}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let s be the semiperimeter of the triangle and let A be its area. Then we have that

$$r=\frac{A}{s},\ r_a=\frac{A}{s-a},\ r_b=\frac{A}{s-b},\ r_c=\frac{A}{s-c},\ R=\frac{abc}{4A}$$

and the inequality can be written as

$$\frac{(s-b)(s-c)}{(s-a)(2s-b-c)}+\frac{(s-a)(s-c)}{(s-b)(2s-a-c)}+\frac{(s-a)(s-b)}{(s-c)(2s-a-b)}\geq 2-\frac{4A^2}{sabc}.$$

Let $x=(b+c-a)/2>0$, $y=(c+a-b)/2>0$, and $z=(a+b-c)/2>0$ then

$$a=x+y,\ b=y+z,\ c=z+x,\ s=x+y+z,\ A^2=xyz(x+y+z)$$

and the inequality becomes

$$\frac{xy}{z(x+y)}+\frac{yz}{x(y+z)}+\frac{zx}{y(z+x)}\geq 2-\frac{4xyz}{(x+y)(y+z)(z+x)}$$

or

$$x^3y^3+z^3x^3+y^3z^3+3x^2y^2z^2\geq xyz^2(yz+zx)+xy^2z(yz+xy)+x^2yz(zx+xy),$$

which holds by Schur's inequality

$$X^3+Z^3+Y^3+3XYZ\geq XY(X+Y)+XZ(X+Z)+YZ(Y+Z)$$

where $X=yz$, $Y=zx$, $Z=xy$. □