Problem 12153
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Proposed by O. Kouba (Syria).

For a real number x whose fractional part is not $1/2$, let $\langle x\rangle$ denote the nearest integer to x. For a positive integer n, let

$$a_n=\sum_{k=1}^{n}\frac{1}{\langle\sqrt{k}\rangle}-2\sqrt{n}.$$

(a) Prove that the sequence $(a_n)_{n\geq 1}$ is convergent, and find its limit L.
(b) Prove that the set $\{\sqrt{n}(a_n-L)\ :\ n\geq 1\}$ is a dense subset of $[0,1/4]$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is easy to verify that

$$\langle\sqrt{k}\rangle=\left\lfloor\sqrt{k}+\frac{1}{2}\right\rfloor=j\quad\text{for } k\in[j(j-1)+1,j(j+1)],$$

and therefore

$$a_n+1=\sum_{k=1}^{n}\frac{1}{\langle\sqrt{k}\rangle}-2\sqrt{n}+1=\frac{n}{\langle\sqrt{n}\rangle}+\langle\sqrt{n}\rangle-1-2\sqrt{n}+1=\frac{(\sqrt{n}-\langle\sqrt{n}\rangle)^2}{\langle\sqrt{n}\rangle}.$$

(a) As n goes to infinity, we have that $\langle\sqrt{n}\rangle\to+\infty$ and

$$0\leq a_n+1=\frac{(\sqrt{n}-\langle\sqrt{n}\rangle)^2}{\langle\sqrt{n}\rangle}=\frac{\left(\{\sqrt{n}+\frac{1}{2}\}-\frac{1}{2}\right)^2}{\langle\sqrt{n}\rangle}\leq\frac{1/4}{\langle\sqrt{n}\rangle}\to 0$$

where $\{x\}=x-\lfloor x\rfloor\in[0,1)$, and we may conclude that $a_n\to L=-1$.

(b) We have that

$$\sqrt{n}(a_n-L)=\sqrt{n}(a_n+1)=\frac{\sqrt{n}(\sqrt{n}-\langle\sqrt{n}\rangle)^2}{\langle\sqrt{n}\rangle}.$$

Let $j=\langle\sqrt{n}\rangle$ then $n\in[j(j-1)+1,j(j+1)]$ and

$$0\leq\sqrt{n}(a_n+1)\leq\frac{\sqrt{j(j+1)}(\sqrt{j(j+1)}-j)^2}{j}=\frac{1}{4}\cdot\frac{\sqrt{j(j+1)}}{\left(\frac{\sqrt{j+1}+\sqrt{j}}{2}\right)^2}<1/4$$

where at the last step we applied the AM-GM inequality.
Let $\alpha\in[0,1/4]$. Then there exist two sequences of integers $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ with $0\leq p_k\leq q_k$ and $q_k\to+\infty$ such that $p_k/q_k\to 2\sqrt{\alpha}\in[0,1]$. Let $n_k=q_k^2+p_k$ then

$$q_k^2\leq n_k\leq q_k^2+q_k<\left(q_k+\frac{1}{2}\right)^2\implies\langle\sqrt{n_k}\rangle=q_k.$$

Therefore, as n goes to infinity,

$$\sqrt{n_k}-\langle\sqrt{n_k}\rangle=\sqrt{n_k}-q_k=\frac{n_k-q_k^2}{\sqrt{n_k}+q_k}=\frac{p_k}{\sqrt{q_k^2+p_k}+q_k}\to\sqrt{\alpha}$$

and

$$\sqrt{n_k}(a_{n_k}+1)=\frac{\sqrt{n_k}(\sqrt{n_k}-\langle\sqrt{n_k}\rangle)^2}{\langle\sqrt{n_k}\rangle}\to\alpha$$

which means that the set $\{\sqrt{n}(a_n+1)\ :\ n\geq 1\}$ is a dense subset of $[0,1/4]$. □