Problem 12149
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Proposed by M. Mehrabi (Sweden).

Prove

$$x^x y^y \left(\Gamma\left(\frac{x+y}{2}\right)\right)^2 \leq \left(\frac{x+y}{2}\right)^{x+y} \Gamma(x)\,\Gamma(y)$$

for all positive real numbers x and y.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$f(x) = \log\left(\frac{\Gamma(x)}{x^x}\right)$$

then the inequality is equivalent to

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

Hence it suffices to show that the second derivative of f is positive which implies that f is convex in $(0, +\infty)$. Recall that

$$\Gamma(x) = \lim_{N\to\infty} \frac{N^x N!}{x\,(x+1)\,(x+2)\cdots(x+N)}$$

and it follows that

$$(\log(\Gamma(x)))'' = \lim_{N\to\infty}\left(x\log(N) + \log(N!) - \sum_{n=0}^{N}\log(x+n)\right)'' = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}.$$

Therefore

$$\begin{aligned} f''(x) = (\log(\Gamma(x)))'' - (x\log(x))'' &= \sum_{n=0}^{\infty}\frac{1}{(x+n)^2} - \frac{1}{x} \\ > \sum_{n=0}^{\infty}\frac{1}{(x+n)(x+n+1)} - \frac{1}{x} &= \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{x+n+1}\right) - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0. \end{aligned}$$

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