Problem 12148
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Proposed by T. Beke (USA).

Let p be a prime number, and let f be a symmetric polynomial in $p-1$ variables with integer coefficients. Suppose that f is homogeneous of degree d and that $p-1$ does not divide d. Prove that p divides $f(1,2,\dots,p-1)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. If $p-1$ does not divide $d \geq 1$ then p is a prime greater than 2 and $\mathbb{Z}_p^*$ is cylic. It follows that if g is a generator then $g^d \not\equiv 1 \pmod{p}$ and the multiplication by g permutes the elements of $\mathbb{Z}_p^*$. Hence, by symmetry and homogeneity,

$$f(1,2,\dots,p-1) \equiv f(g\cdot 1, g\cdot 2, \dots, g\cdot(p-1)) = g^d \cdot f(1,2,\dots,p-1) \pmod{p}$$

which implies that

$$(g^d-1)f(1,2,\dots,p-1) \equiv 0 \pmod{p}.$$

Since $g^d \not\equiv 1 \pmod{p}$, we have that p divides $f(1,2,\dots,p-1)$. □