Problem 12147
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Let $ABCD$ be a quadrilateral that is not a parallelogram. The Newton line of $ABCD$ is the line that connects the midpoints of the diagonals AC and BD. Let X be the intersection of the perpendicular bisectors of AB and CD, and let Y be the intersection of the perpendicular bisectors of BC and DA. Prove that XY is perpendicular to the Newton line of $ABCD$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. A vector approach. Since X is the intersection of the perpendicular bisectors of AB and CD we have that

$$\left(X-\frac{A+B}{2}\right)\cdot(A-B)=0 \quad\text{and}\quad \left(X-\frac{C+D}{2}\right)\cdot(C-D)=0$$

where $\cdot$ denotes the scalar product. Similarly, since Y be the intersection of the perpendicular bisectors of BC and DA, we have that

$$\left(Y-\frac{B+C}{2}\right)\cdot(B-C)=0 \quad\text{and}\quad \left(Y-\frac{D+A}{2}\right)\cdot(D-A)=0.$$

Hence, by taking the sum of the above four equations, we find

$$\left(X-\frac{A+B}{2}\right)\cdot(A-B)+\left(X-\frac{C+D}{2}\right)\cdot(C-D)+\left(Y-\frac{B+C}{2}\right)\cdot(B-C)+\left(Y-\frac{D+A}{2}\right)\cdot(D-A)=0$$

which reduces to

$$2(X-Y)\cdot\left(\frac{A+C}{2}-\frac{B+D}{2}\right)=0$$

and we may conclude that the line XY is perpendicular to the line that connects $\frac{A+C}{2}$, the midpoint of the diagonal AC, and $\frac{B+D}{2}$, the midpoint of the diagonal BD, that is the Newton line of $ABCD$.
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