Problem 12144
(American Mathematical Monthly, Vol.126, November 2019)

Proposed by Nguyen Minh Ha and Tran Quang Hung (Vietnam).

Let $MNPQ$ be a square inscribed in quadrilateral $ABCD$ with M, N, P, and Q lying on sides AB, BC, CD, DA, respectively. Let W, X, Y, and Z be the points where the incircles of triangles AQM, BMN, CNP, and DPQ touch QM, MN, NP, and PQ, respectively. Prove that $ABCD$ has an inscribed circle if and only if WY is perpendicular to XZ.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let l be the length of the side of the square $MNPQ$. Since W, X, Y, and Z are the points where the incircles of triangles AQM, BMN, CNP, and DPQ touch QM, MN, NP, and PQ, respectively, it follows that

$$2|PW| = |PD| + l - |DQ|, \qquad 2|QX| = |QA| + l - |AM|,$$
$$2|MY| = |MB| + l - |BN|, \qquad 2|NZ| = |NC| + l - |CP|.$$

Now WY is perpendicular to XZ if and only if

$$|PW| + |MY| = |QX| + |NZ|$$

that is

$$|PD| + l - |DQ| + |MB| + l - |BN| = |QA| + l - |AM| + |NC| + l - |CP|$$

which can be written as

$$|AM| + |MB| + |CP| + |PD| = |BN| + |NC| + |DQ| + |QA|$$

and therefore (M, N, P, and Q lie on sides AB, BC, CD, DA, respectively)

$$|AB| + |CD| = |BC| + |DA|$$

which holds if and only if $ABCD$ has an inscribed circle.