Problem 12143
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Proposed by J. A. Scott (UK).

Compute

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show that the limit exists and it is equal to $\frac{e}{e-1}$.
Lower bound. Since

$$\left(\frac{k}{n}\right)^{k}=\left(1+\frac{n-k}{k}\right)^{-k}\geq e^{-(n-k)}$$

then

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}\geq\sum_{k=1}^{n}e^{n-k}=\sum_{j=1}^{n-1}e^{-j}.$$

and

$$\liminf_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}\geq\sum_{j=0}^{\infty}e^{-j}=\frac{e}{e-1}. \tag{1}$$

Upper bound. Let $t\in(0,1)$. Then we split the given sum as

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}=\sum_{1\leq k\leq tn}\left(\frac{k}{n}\right)^{k}+\sum_{tn<k<n-n^{1/3}}\left(\frac{k}{n}\right)^{k}+\sum_{n-n^{1/3}\leq k\leq n}\left(\frac{k}{n}\right)^{k}$$

It follows that

$$\sum_{1\leq k\leq tn}\left(\frac{k}{n}\right)^{k}\leq\sum_{1\leq k\leq tn}t^{k}\leq\sum_{k=1}^{\infty}t^{k}=\frac{t}{1-t}.$$

Moreover

$$\sum_{tn<k<n-n^{1/3}}\left(\frac{k}{n}\right)^{k}\leq\sum_{t<k/n<1-1/n^{2/3}}\left(\frac{k}{n}\right)^{tn}\leq n\int_{0}^{1-1/n^{2/3}}x^{tn}\,dx=\frac{n}{tn+1}\left(1-\frac{1}{n^{2/3}}\right)^{tn+1}$$

and

$$\begin{aligned}\sum_{n-n^{1/3}\leq k\leq n}\left(\frac{k}{n}\right)^{k}&=\sum_{0\leq j\leq n^{1/3}}\left(1-\frac{j}{n}\right)^{n-j}=\sum_{0\leq j\leq n^{1/3}}\exp\left((n-j)\ln\left(1-\frac{j}{n}\right)\right)\\&\leq\sum_{0\leq j\leq n^{1/3}}\exp\left((n-j)\left(-\frac{j}{n}\right)\right)=\sum_{0\leq j\leq n^{1/3}}e^{-j}\cdot e^{j^2/n}\leq e^{1/n^{1/3}}\sum_{j=0}^{n-1}e^{-j}.\end{aligned}$$

Therefore

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}\leq\frac{t}{1-t}+\frac{n}{tn+1}\left(1-\frac{1}{n^{2/3}}\right)^{tn+1}+e^{1/n^{1/3}}\sum_{j=0}^{n-1}e^{-j}$$

and

$$\limsup_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}\leq\frac{t}{1-t}+0+\frac{e}{e-1}.$$

By letting $t\to 0^{+}$, we find

$$\limsup_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{k}\leq\frac{e}{e-1}. \tag{2}$$

Finally, by (1) and (2), we may conclude that our initial claim holds. □