Problem 12142
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Proposed by C. Chiser (Romania).

Let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function satisfying $\int_a^b f(x)\,dx = 0$. Prove

$$\int_a^b (f''(x))^2\,dx \geq \frac{980}{(8\sqrt{2}-1)^2} \cdot \frac{(f(a)+f(b))^2}{(b-a)^3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. After applying integration by parts twice, we find

$$\begin{aligned}\int_a^b (b-x)(x-a)f''(x)\,dx &= [(b-x)(x-a)f'(x)]_a^b - \int_a^b (b+a-2x)f'(x)\,dx \\ &= 0 - [(b+a-2x)f'(x)]_a^b - 2\int_a^b f(x)\,dx \\ &= (b-a)(f(a)+f(b)) + 0.\end{aligned}$$

On the other hand, by Cauchy-Schwarz inequality

$$\left(\int_a^b (b-x)(x-a)f''(x)\,dx\right)^2 \leq \int_a^b (b-x)^2(x-a)^2\,dx \int_a^b (f''(x))^2\,dx = \frac{(b-a)^5}{30}\int_a^b (f''(x))^2\,dx.$$

It follows that

$$\int_a^b (f''(x))^2\,dx \geq 30 \cdot \frac{(f(a)+f(b))^2}{(b-a)^3}$$

and we are done since $30 > \dfrac{980}{(8\sqrt{2}-1)^2} \approx 9.21$.
Note that 30 is the best constant because for the polynomial

$$f(x) = 10(x-a)^3(2b-a-x) - 3(b-a)^4$$

we have $\int_a^b f(x)\,dx = 0$, $f(a)+f(b) = 4(b-a)^4 \neq 0$ and

$$\frac{(b-a)^3}{(f(a)+f(b))^2}\int_a^b (f''(x))^2\,dx = 30.$$

□