Problem 12141
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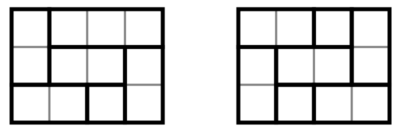
Proposed by R. Tauraso (Italy).

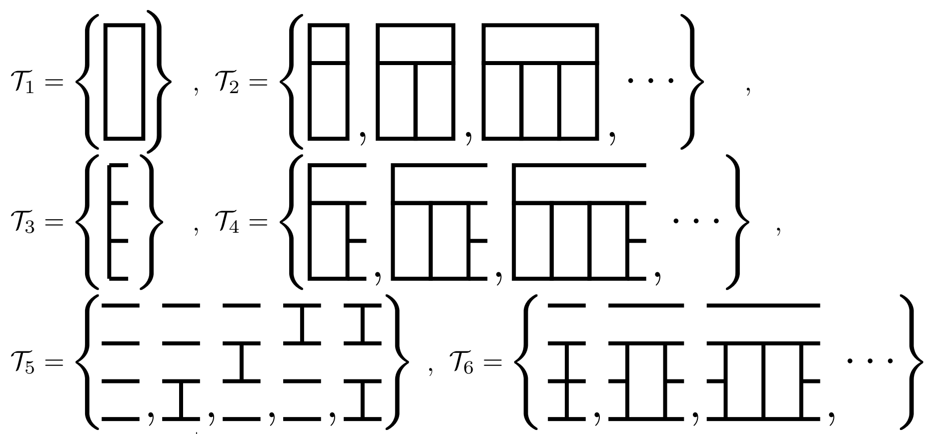
Consider tilings of a rectangle with tiles each of which is 1-by-k for some positive integer k. A fault line through a tiling is a horizontal or vertical line through the tiling that cuts through no tile. A tiling is fault-free if it has no fault lines. Two of the 22 fault-free tilings of the 3-by-4 rectangle are shown below.



Let f_n be the number of fault-free tilings of a 3-by-n rectangle.
Find **(a)** $\lim_{n\to\infty} \sqrt[n]{f_n}$, *and* **(b)** $\lim_{n\to\infty} \sqrt[n]{f_{n+1}f_{n-1} - f_n^2}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Consider the following sets of configurations



For any $\mathcal{T}_i$, let $\overline{\mathcal{T}_i}$ and $\mathcal{T}_i^{|}$ be the sets where the configurations are reflected across the horizontal middle axis or the vertical middle axis, respectively. The combinatorial class of the $(3 \times n)$-tilings for $n \geq 1$ which are vertically fault-free, is given by

$$\mathcal{T}_1 + \mathcal{T}_2 + \overline{\mathcal{T}_2} + \left(\mathcal{T}_3 + \mathcal{T}_4 + \overline{\mathcal{T}_4}\right) \times \mathsf{Seq}\left(T_5 \cup T_6 \cup \overline{T_6}\right) \times \left(\mathcal{T}_3^{|} + \mathcal{T}_4^{|} + \overline{\mathcal{T}_4}^{|}\right).$$

Hence the corresponding generating function is

$$G(z) = z + \frac{2z}{1-z} + \frac{z\left(1 + \frac{2z}{1-z}\right)^2}{1 - 5z - \frac{2z}{1-z}}.$$

Note that for $n \geq 2$ a vertically fault-free tiling is also horizontally fault-free, and therefore simply fault-free, if and only if it contains at least two vertical dominoes one touching the bottom horizontal borderline and the other the top horizontal borderline. So, in order to eliminate the tilings which have a horizontal fault-line, we determine three more generating functions.
By using the same approach as before, we have that the g.f. of the number of vertically fault-free tilings for $n \geq 2$ with no vertical domino touching the top borderline is

$$G_t(z) = \frac{z}{1-z} + \frac{z\left(1 + \frac{z}{1-z}\right)^2}{1 - 6z - \frac{z}{1-z}}.$$

By symmetry, this is also the g.f. G_b of the number of vertically fault-free tilings for $n \geq 2$ with no vertical domino touching the bottom borderline.

Finally

$$G_0(z) = \frac{z}{1-7z}$$

is the g.f. of the number of vertically fault-free tilings for $n \geq 2$ with no vertical domino at all. Hence the g.f. of the number of fault-free tilings for $n \geq 1$ is

$$\begin{aligned} F(z) = \sum_{n \geq 1} f_n z^n &= G(z) - G_b(z) - G_t(z) + G_0(z) \\ &= z + \frac{2z^3(1-6z)^2}{(1-7z)(1-8z+6z^2)(1-8z+5z^2)} \\ &= z + 2z^3 + 22z^4 + 204z^5 + 1804z^6 + 15538z^7 + 131286z^8 + \dots . \end{aligned}$$

The poles of the rational functions F are real and simple:

$$\frac{1}{4+\sqrt{11}} < \frac{1}{4+\sqrt{10}} < \frac{1}{7} < \frac{1}{4-\sqrt{10}} < \frac{1}{4-\sqrt{11}}.$$

It follows that there are some non-zero real constants A, B, C such that

$$f_n = A\,(4+\sqrt{11})^n + B\,(4+\sqrt{10})^n + C\,7^n + o(1).$$

From this we may conclude that, as n goes to infinity,

$$\lim_{n\to\infty} \sqrt[n]{f_n} = 4 + \sqrt{11}$$

and

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{f_{n+1}f_{n-1} - f_n^2} &= \lim_{n\to\infty} \sqrt[n]{AB(\sqrt{11}-\sqrt{10})^2 \left((4+\sqrt{11})(4+\sqrt{10})\right)^{n-1}} \\ &= (4+\sqrt{11})(4+\sqrt{10}). \end{aligned}$$

□