Problem 12136
(American Mathematical Monthly, Vol.126, October 2019)

Proposed by A. Stadler (Switzerland).

Prove

$$a^2 + b^2 + c^2 \geq a\sqrt[4]{\frac{b^4 + c^4}{2}} + b\sqrt[4]{\frac{c^4 + a^4}{2}} + c\sqrt[4]{\frac{a^4 + b^4}{2}}$$

for all positive real numbers a, b, and c.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $2(x^2 - xy + y^2)^2 - (x^4 + y^4) = (x - y)^4 \geq 0$, it follows that

$$\sqrt{x^2 - xy + y^2} \geq \sqrt[4]{\frac{x^4 + y^4}{2}}$$

for all real x, y and it suffices to show the stronger inequality

$$a^2 + b^2 + c^2 \geq a\sqrt{b^2 - bc + c^2} + b\sqrt{c^2 - ca + a^2} + c\sqrt{a^2 - ab + b^2}.$$

Moreover, the concavity of $x \to f(x) = \sqrt{x}$ implies that

$$f\left(\frac{a(b^2 - bc + c^2) + b(c^2 - ca + a^2) + c(a^2 - ab + b^2)}{a + b + c}\right) \geq \frac{af(b^2 - bc + c^2) + bf(c^2 - ca + a^2) + cf(a^2 - ab + b^2)}{a + b + c}$$

that is

$$\sqrt{a + b + c}\sqrt{\sum_{\text{sym}} ab^2 - 3abc} \geq a\sqrt{b^2 - bc + c^2} + b\sqrt{c^2 - ca + a^2} + c\sqrt{a^2 - ab + b^2}$$

Hence it remains to show that

$$(a^2 + b^2 + c^2)^2 \geq (a + b + c)\left(\sum_{\text{sym}} ab^2 - 3abc\right)$$

that is

$$a^2(a - b)(a - c) + b^2(b - a)(b - c) + c^2(c - a)(c - b) \geq 0$$

which holds by Schur's inequality. $\square$

Remark: the given inequality holds for all real numbers a, b, c. It does not hold if we replace $\sqrt[4]{\frac{(\cdot)^4+(\cdot)^4}{2}}$ with $\sqrt[n]{\frac{(\cdot)^n+(\cdot)^n}{2}}$ with any integer $n \geq 5$. Take $a = 1/2$, $b = c = 1$.