Problem 12134
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Proposed by P. Bracken (USA).

Evaluate the series

$$\sum_{n=1}^{\infty}\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1-\frac{1}{2n}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By Stolz-Cesaro,

$$\lim_{N\to\infty}\frac{\sum_{k=N}^{\infty}1/k^2-1/N}{1/N^2}\lim_{N\to\infty}\frac{-1/N^2-1/(N+1)+1/N}{1/(N+1)^2-1/N^2}=\frac{1}{2},$$

and therefore

$$\sum_{k=N}^{\infty}\frac{1}{k^2}=\frac{1}{N}+\frac{1}{2N^2}+o(1/N^2).$$

Hence, for $N\geq 1$, we have that

$$\begin{aligned}
\sum_{n=1}^{N}\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1-\frac{1}{2n}\right)&=\sum_{n=1}^{N}n\sum_{k=n}^{N}\frac{1}{k^2}+\sum_{n=1}^{N}n\sum_{k=N+1}^{\infty}\frac{1}{k^2}-N-\frac{H_N}{2}\\
&=\sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=1}^{k}n+\frac{N(N+1)}{2}\sum_{k=N+1}^{\infty}\frac{1}{k^2}-N-\frac{H_N}{2}\\
&=\sum_{k=1}^{N}\frac{k(k+1)}{2k^2}+\frac{N(N+1)}{2}\left(\frac{1}{N}+\frac{1}{2N^2}+o(1/N^2)-\frac{1}{N^2}\right)-N-\frac{H_N}{2}\\
&=\frac{N}{2}+\frac{H_N}{2}+\frac{N+1}{2}-\frac{N+1}{4N}+o(1)-N-\frac{H_N}{2}\\
&=\frac{1}{4}+o(1)
\end{aligned}$$

where $H_N=\sum_{k=1}^{N}1/k$. Finally, as $N\to\infty$, we find that the required sum is equal to $1/4$. $\square$