Problem 12132
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Let n be a positive integer, and let $X_0 = n+1$. Repeatedly choose the integer X_k uniformly at random among the integers j with $1 \leq j < X_{k-1}$, stopping when $X_m = 1$.
(a) What is the expected value of m?
(b) What is the expected value of X_{m-1}?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. (a) $E[m(n)]$, the expected value of $m(n)$, is equal to 1 for $n = 1$. For $n > 1$, X_1 takes a value $k \in \{1, 2, \ldots, n\}$ with probability $1/n$ and therefore the stopping time $m(n)$ is $m(k) + 1$ for $k \geq 2$ and it is 1 for $k = 1$. Hence, we have the recurrence

$$E[m(n)] = \frac{1}{n}\left(\sum_{k=2}^{n}(E[m(k-1)]+1)+1\right).$$

We show that $E[m(n)] = H_n = \sum_{j=1}^{n}\frac{1}{j}$ by verifying the above recurrence,

$$\frac{1}{n}\left(\sum_{k=1}^{n-1}(H_k+1)+1\right) = 1+\frac{1}{n}\sum_{k=1}^{n-1}H_k = 1+\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{j}\sum_{k=j}^{n-1}1 = 1+\frac{1}{n}\sum_{j=1}^{n-1}\frac{n-j}{j} = H_n.$$

(b) $E[X_{m-1}(n)]$, the expected value of $X_{m-1}(n)$, is equal to 2 for $n = 1$. For $n > 1$, X_1 takes a value $k \in \{1, 2, \ldots, n\}$ with probability $1/n$ and therefore $X_{m-1}(n)$ is $X_{m-1}(k-1)$ for $k \geq 2$ and it is $n+1$ for $k = 1$. Hence, we have the recurrence

$$E[X_{m-1}(n)] = \frac{1}{n}\left(\sum_{k=2}^{n}E[X_{m-1}(k-1)]+(n+1)\right).$$

As before, it is easy to verify that $E[X_{m-1}(n)] = H_n + 1$,

$$\frac{1}{n}\left(\sum_{k=1}^{n-1}(H_k+1)+(n+1)\right) = 2+\frac{1}{n}\sum_{k=1}^{n-1}H_k = H_n+1.$$