Problem 12129
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Compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{2+\cdots}}}}}$$

where the sequence of signs consists of $n-1$ plus signs followed by a minus sign and repeats with period n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $f_n(x)=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2-x}}}}$. For $x_0\in[0,2]$, we define the recurrence

$$x_k=f_n(x_{k-1})\quad \text{for } k\geq 1.$$

We will show that

$$\lim_{k\to+\infty} x_k = 2\cos\left(\frac{\pi}{2^n+1}\right).$$

Let $\alpha_k=\arccos(x_k/2)\in[0,\pi/2]$ then

$$2\cos(\alpha_k)=x_k=f_n(x_{k-1})=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2-2\cos(\alpha_{k-1})}}}}$$

is equivalent to

$$2\cos(2\alpha_k)=4\cos^2(\alpha_k)-2=\sqrt{2+\sqrt{2+\cdots+\sqrt{2-2\cos(\alpha_{k-1})}}}$$

and after $n-1$ steps we get

$$2\cos(2^{n-1}\alpha_k)=\sqrt{2-2\cos(\alpha_{k-1})}=2\sin(\alpha_{k-1}/2)=2\cos(\pi/2-\alpha_{k-1}/2)$$

which implies that

$$\alpha_k=\frac{\pi-\alpha_{k-1}}{2^n}.$$

Hence, as $k\to+\infty$,

$$\alpha_k=\frac{\pi}{2^n}-\frac{\alpha_{k-1}}{2^n}=\frac{\pi}{2^n}-\frac{\pi}{2^{2n}}+\frac{\alpha_{k-2}}{2^{2n}}=\pi\sum_{j=1}^{k}\frac{(-1)^{j-1}}{2^{jn}}+\frac{(-1)^k\alpha_0}{2^{kn}}\to\frac{\pi}{2^n+1},$$

and therefore

$$x_k=2\cos(\alpha_k)\to 2\cos\left(\frac{\pi}{2^n+1}\right).$$

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