Problem 12128
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Let F_n be the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Find, in terms of n, the number of trailing zeros in the decimal representation of F_n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $\nu_p(n)$ be the highest power of a prime p which divides a positive integer n. We will show that the number of trailing zeros of F_n is equal to

$$\min(\nu_2(F_n), \nu_5(F_n)) = \begin{cases} 0 & \text{if } b = 0 \text{ or } c = 0, \\ 1 & \text{if } b \geq 1 \text{ and } c \geq 1 \text{ and } a = 0, \\ \min(a+2, c) & \text{if } b \geq 1 \text{ and } c \geq 1 \text{ and } a \geq 1. \end{cases}$$

where $n = 2^a \cdot 3^b \cdot 5^c \cdot m$ with $a, b, c \geq 0$ and $\gcd(m, 30) = 1$.

The above formula follows straightforwardly from the next two claims.

Claim 1. $\nu_5(F_n) = \nu_5(n)$.
For $n \geq 1$, by the Binet's formula,

$$2^{n-1} F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}} = n + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} 5^k.$$

Since for $1 \leq k \leq \lfloor (n-1)/2 \rfloor$,

$$\begin{aligned} \nu_5\left(\binom{n}{2k+1} 5^k\right) &= \nu_5\left(\frac{n}{2k+1}\binom{n-1}{2k-1} 5^k\right) = \nu_5(n) - \nu_5(2k+1) + \nu_5\left(\binom{n-1}{2k-1} 5^k\right) \\ &\geq \nu_5(n) - \nu_5(2k+1) + k > \nu_5(n) \end{aligned}$$

it follows that $\nu_5(F_n) = \nu_5(2^{n-1} F_n) = \nu_5(n)$.

Claim 2. $\nu_2(F_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 3 \pmod 6, \\ \nu_2(n) + 2 & \text{if } n \equiv 0 \pmod 6. \end{cases}$

It is easy to see by induction that F_n is even if and only if $n \equiv 0 \pmod 3$.
For $m \geq 1$, by the Binet's formula,

$$F_{3m} = \frac{(1+\sqrt{5})^{3m} - (1-\sqrt{5})^{3m}}{2^{3m}\sqrt{5}} = \frac{(2+\sqrt{5})^m - (2-\sqrt{5})^m}{\sqrt{5}} = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2k+1} 5^k 2^{m-2k}.$$

Thus, if $n = 3m = 6q + 3$ then

$$F_n = 2 \cdot 5^q + 8 \sum_{k=0}^{q-1} \binom{2q+1}{2k+1} 5^k 4^{q-1-k}$$

and we find that $\nu_2(F_n) = 1$.
If $n = 3m = 6q$ then

$$F_n = 4(2q)5^{q-1} + \sum_{k=0}^{q-2} \binom{2q}{2k+1} 5^k 4^{q-k}.$$

Therefore $\nu_2(F_n) = \nu(4(2q)5^{q-1}) = \nu_2(2q) + 2 = \nu_2(n) + 2$ because for $0 \leq k \leq q-2$,

$$\begin{aligned} \nu_2\left(\binom{2q}{2k+1} 5^k 4^{q-k}\right) &= \nu_5\left(\frac{2q}{2k+1}\binom{2q-1}{2k} 5^k 4^{q-k}\right) \\ &= \nu_2(2q) + \nu_2\left(\binom{2q-1}{2k} 5^k\right) + 2(q-k) > \nu_2(2q) + 2. \end{aligned}$$

$\square$