Problem 12127
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Calculate

$$\int_0^1 \left(\frac{\mathrm{Li}_2(1)-\mathrm{Li}_2(x)}{1-x}\right)^2 dx.$$

where $\mathrm{Li}_2(z)=\sum_{k=1}^{\infty} z^k/k^2$ *is the dilogarithm function.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $\mathrm{Li}_2(1)-\mathrm{Li}_2(x)=\mathrm{Li}_2(1-x)+\ln(x)\ln(1-x)$, the given integral I can be written as

$$\begin{aligned}I&=\int_0^1\left(\frac{\mathrm{Li}_2(1-x)+\ln(x)\ln(1-x)}{1-x}\right)^2 dx=\int_0^1\left(\frac{\mathrm{Li}_2(x)+\ln(x)\ln(1-x)}{x}\right)^2 dx\\&=\int_0^1\frac{\mathrm{Li}_2^2(x)}{x^2}\,dx+2\int_0^1\frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x^2}\,dx+\int_0^1\frac{\ln^2(x)\ln^2(1-x)}{x^2}\,dx.\end{aligned}$$

The first integral:

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2^2(x)}{x^2}\,dx&=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-2}\mathrm{Li}_2(x)\,dx=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^1 x^{n+k-2}\,dx\\&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2k^2(n+k-1)}\\&=\zeta(3)+\zeta(2)\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)}-\sum_{n=2}^{\infty}\frac{H_{n-1}}{n^2(n-1)^2}.\end{aligned}$$

The second integral:

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x^2}\,dx&=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-2}\ln(x)\ln(1-x)\,dx=-\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{n+k-2}\ln(x))\,dx\\&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2k(n+k-1)^2}\\&=\zeta(3)-\zeta(2)\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)}+\sum_{n=2}^{\infty}\frac{H_{n-1}^{(2)}}{n^2(n-1)}+\sum_{n=2}^{\infty}\frac{H_{n-1}}{n^2(n-1)^2}.\end{aligned}$$

The third integral:

$$\int_0^1\frac{\ln^2(x)\ln^2(1-x)}{x^2}\,dx=2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^1 x^{n-1}\ln^2(x)\,dx=4\sum_{n=1}^{\infty}\frac{H_n}{(n+1)n^3}.$$

Hence, by partial fraction decomposition,

$$\begin{aligned}I&=3\zeta(3)-\zeta(2)\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)}+2\sum_{n=2}^{\infty}\frac{H_{n-1}^{(2)}}{n^2(n-1)}+\sum_{n=2}^{\infty}\frac{H_{n-1}}{n^2(n-1)^2}+4\sum_{n=1}^{\infty}\frac{H_n}{(n+1)n^3}\\&=3\zeta(3)-\zeta(2)(2-\zeta(2))+(2\zeta(3)-3\zeta(4)/2)+(-2\zeta(2)+3\zeta(3))+(4\zeta(2)-8\zeta(3)+5\zeta(4))\\&=\zeta(2)^2+\frac{7\zeta(4)}{2}=\frac{\pi^4}{15}\end{aligned}$$

where we used the known multiple zeta values

$$\sum_{n=2}^{\infty}\frac{H_{n-1}}{n^2}=\zeta(2,1)=\zeta(3),\quad \sum_{n=2}^{\infty}\frac{H_{n-1}}{n^3}=\zeta(3,1)=\frac{\zeta(4)}{4},\quad \sum_{n=2}^{\infty}\frac{H_{n-1}^{(2)}}{n^2}=\zeta(2,2)=\frac{3\zeta(4)}{4}.$$

□