Problem 12126
(American Mathematical Monthly, Vol.126, August-September 2019)

Proposed by M. Tetiva (Romania).

Let $P(n)$ be the greatest prime divisor of the positive integer n. Prove that $P(n^2-n+1) < P(n^2+n+1)$ and $P(n^2-n+1) > P(n^2+n+1)$ each hold for infinitely many positive integers n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $q_n := P(n^2+n+1)$, then, since $n^2-n+1 = (n-1)^2+(n-1)+1$ it follows that $P(n^2-n+1) = q_{n-1}$. Moreover,

$$\gcd(n^2-n+1, n^2+n+1) = \gcd(n^2-n+1, 2n) = \gcd(n^2-n+1, n) = \gcd(1, n) = 1.$$

which implies that $q_n \neq q_{n-1}$.

Assume that the claim is false, then, by the previous remark, the sequence of primes

$$(q_n)_{n\geq 1} = 3, 7, 13, 7, 31, 43, 19, 73, 13, 37, 19, 157, 61, 211, 241, \ldots$$

should be eventually strictly decreasing or strictly increasing, but this is impossible because

$$(n^2)^2 + (n^2) + 1 = (n^2+n+1)(n^2-n+1)$$

and therefore $q_{n^2} = \max(q_n, q_{n-1})$ which yields a contradiction since $n^2 > n > n-1$ for $n > 1$. $\square$