Problem 12124
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Proposed by M. Omarjee (France).

Let $p > 1$, and let $a_1, a_2, \dots$ be a sequence of positive real numbers. Prove that if

$$\sum_{k=1}^{n} \frac{1}{1+a_k^p} = O\left(\frac{1}{1+a_n^p}\right),$$

then

$$\sum_{k=1}^{n} \frac{1}{1+a_k} = O\left(\frac{1}{(1+a_n^p)^{1/p}}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will prove a more general result.

Let $p > 0$ and let $(x_n)_n$, $(y_n)_n$ be two sequences of positive real numbers, such that $y_n^p \leq x_n$ for all $n \geq 1$. Then

$$\sum_{k=1}^{n} x_k = O(x_n) \implies \sum_{k=1}^{n} y_k = O\left(x_n^{1/p}\right).$$

If $p > 1$, $x_n = \frac{1}{1+a_n^p}$ and $y_n = \frac{1}{1+a_n}$, then $y_n^p \leq x_n$ if and only if $1 + a_n^p \leq (1+a_n)^p$ which holds and we are done.

By definition, $\sum_{k=1}^{n} x_k = O(x_n)$ implies that there is $C > 1$ such that for $n \geq 1$,

$$S_n := \sum_{k=1}^{n} x_k < Cx_n = C(S_n - S_{n-1}) \implies S_{n-1} < qS_n \quad \text{where } q = 1 - \frac{1}{C} \in (0,1).$$

Hence , for $1 \leq k \leq n$,

$$y_k^p \leq x_k \leq S_k \leq qS_{k+1} \leq \cdots \leq q^{n-k} S_n < q^{n-k} C x_n \implies y_k < (q^{n-k} C x_n)^{1/p}.$$

Finally

$$\sum_{k=1}^{n} y_k < (Cx_n)^{1/p} \sum_{k=1}^{n} q^{(n-k)/p} < (Cx_n)^{1/p} \sum_{k=0}^{\infty} q^{k/p} = \frac{C^{1/p}}{1-q^{1/p}} x_n^{1/p}$$

that is $\sum_{k=1}^{n} y_k = O\left(x_n^{1/p}\right)$. □