Problem 12120
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Proposed by M. Bataille (France).

For positive integers n and k with $n \geq k$, let $a(n,k) = \sum_{j=0}^{k-1} \binom{n}{j} 3^j$.
(a) Evaluate

$$\lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{n} \frac{a(n,k)}{k}.$$

(b) Evaluate

$$\lim_{n\to\infty} n \left(4^n L - \sum_{k=1}^{n} \frac{a(n,k)}{k} \right)$$

where L is the limit in part (a).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will prove a more general result: for $x > 0$ let $A_n(x) := \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k-1} \binom{n}{j} x^j$ then

$$\lim_{n\to\infty} \frac{A_n(x)}{(x+1)^n} = \ln\left(1 + \frac{1}{x}\right) \quad \text{and} \quad \lim_{n\to\infty} n \left((x+1)^n \ln\left(1+\frac{1}{x}\right) - A_n(x) \right) = \frac{1}{x}.$$

Therefore, for $x = 3$, the limits are: (a) $\ln(4/3)$ and (b) $1/3$.
We first show that the following recurrence holds

$$A_0(x) = 0\,, \qquad A_n(x) = (x+1)A_{n-1}(x) + \frac{1}{n} \quad \text{for } n \geq 1.$$

We have that

$$\begin{aligned}
A_n(x) - (x+1)A_{n-1}(x) &= \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k-1} \binom{n}{j} x^j - (x+1) \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=0}^{k-1} \binom{n-1}{j} x^j \\
&= \frac{(x+1)^n - x^n}{n} + \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{k-1} \binom{n-1}{j-1} x^j - \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=0}^{k-1} \binom{n-1}{j} x^{j+1} \\
&= \frac{(x+1)^n - x^n}{n} + \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{k-1} \binom{n-1}{j-1} x^j - \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{k} \binom{n-1}{j-1} x^j \\
&= \frac{(x+1)^n - x^n}{n} - \sum_{k=1}^{n-1} \frac{1}{k} \binom{n-1}{k-1} x^k = \frac{(x+1)^n - x^n}{n} - \frac{1}{n} \sum_{k=1}^{n-1} \binom{n}{k} x^k \\
&= \frac{(x+1)^n - x^n}{n} - \frac{(1+x)^n - x^n - 1}{n} = \frac{1}{n}.
\end{aligned}$$

By the above recurrence, it follows that $A_n(x) = (x+1)^n \sum_{k=1}^{n} \frac{1}{k(x+1)^k}$ and the first limit is

$$\lim_{n\to\infty} \frac{A_n(x)}{(x+1)^n} = \sum_{k=1}^{\infty} \frac{1}{k(x+1)^k} = \ln\left(\frac{1}{1 - \frac{1}{x+1}} \right) = \ln\left(1 + \frac{1}{x}\right).$$

Finally, for the last limit, we use the Stolz-Cesaro Theorem,

$$\begin{aligned}
\lim_{n\to\infty} n \left((x+1)^n \ln\left(1+\frac{1}{x}\right) - A_n(x) \right) &= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{x}\right) - \sum_{k=1}^{n} \frac{1}{k(x+1)^k}}{\frac{1}{n(x+1)^n}} \\
&\overset{SC}{=} \lim_{n\to\infty} \frac{-\frac{1}{(n+1)(x+1)^{n+1}}}{\frac{1}{(n+1)(x+1)^{n+1}} - \frac{1}{n(x+1)^n}} \\
&= \lim_{n\to\infty} \frac{1}{-1 + \frac{n+1}{n}(x+1)} = \frac{1}{x}.
\end{aligned}$$

□