Problem 12118
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Proposed by H. Ohtsuka (Japan).

Compute

$$\sum_{n=0}^{\infty} \frac{1}{F_{2mn} + iF_m}$$

where m is an odd integer and F_n is the n-th Fibonacci number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By using the known identity

$$F_{i+j}F_{i+k} - F_iF_{i+j+k} = (-1)^i F_j F_k$$

with $i, j, k \in \mathbb{Z}$, it follows that for any odd integer m, $F_m = F_{-m}$ and

$$F_{2nm+m}F_{2nm-m} - F_{2nm}^2 = F_m^2 \implies F_{(2n+1)m}F_{(2n-1)m} = F_{2nm}^2 + F_m^2$$
$$F_mF_{2nm-m} - F_{-m}F_{2nm+m} = -F_{2m}F_{2nm} \implies \frac{F_{2nm}}{F_{(2n+1)m}F_{(2n-1)m}} = \frac{F_m/F_{2m}}{F_{(2n-1)m}} - \frac{F_m/F_{2m}}{F_{(2n+1)m}}$$
$$F_{2nm+m}F_{2nm-2m} - F_{2nm}F_{2nm-m} = -F_mF_{2m} \implies \frac{F_m}{F_{(2n+1)m}F_{(2n-1)m}} = \frac{F_{2nm}/F_{2m}}{F_{(2n+1)m}} - \frac{F_{(2n-2)m}/F_{2m}}{F_{(2n-1)m}}.$$

Therefore,

$$\begin{aligned}\frac{1}{F_{2mn} + iF_m} &= \frac{F_{2nm} - iF_m}{F_{2nm}^2 + F_m^2} = \frac{F_{2mn} - iF_m}{F_{(2n+1)m}F_{(2n-1)m}} \\ &= \frac{F_m}{F_{2m}}\left(\frac{1}{F_{(2n-1)m}} - \frac{1}{F_{(2n+1)m}}\right) - \frac{i}{F_{2m}}\left(\frac{F_{2nm}}{F_{(2n+1)m}} - \frac{F_{(2n-2)m}}{F_{(2n-1)m}}\right)\end{aligned}$$

Hence the given sum is telescopic:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{F_{2mn} + iF_m} &= \frac{F_m}{F_{2m}} \sum_{n=0}^{\infty}\left(\frac{1}{F_{(2n-1)m}} - \frac{1}{F_{(2n+1)m}}\right) - \frac{i}{F_{2m}} \sum_{n=0}^{\infty}\left(\frac{F_{2nm}}{F_{(2n+1)m}} - \frac{F_{(2n-2)m}}{F_{(2n-1)m}}\right) \\ &= \frac{F_m}{F_{2m}F_{-m}} - \frac{i}{F_{2m}}\left(\phi^{-m} - \frac{F_{-2m}}{F_{-m}}\right) \\ &= \frac{1}{F_{2m}} - \frac{i}{F_{2m}}\left(\phi^{-m} + \frac{F_{2m}}{F_m}\right) = \frac{1 - i\phi^m}{F_{2m}}\end{aligned}$$

where $\phi = (1+\sqrt{5})/2$ and by the Binet's formula

$$\lim_{n\to\infty} \frac{F_{2nm}}{F_{(2n+1)m}} = \phi^{-m} \quad \text{and} \quad \frac{F_{2m}}{F_m} = \phi^m - \phi^{-m}.$$

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