Problem 12117
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Proposed by M. Bataille (France).

Let n be a nonnegative integer. Prove that

$$\frac{\sin^{n+1}(4\pi/7)}{\sin^{n+2}(\pi/7)} - \frac{\sin^{n+1}(\pi/7)}{\sin^{n+2}(2\pi/7)} + (-1)^n \frac{\sin^{n+1}(2\pi/7)}{\sin^{n+2}(4\pi/7)}$$

is equal to

$$2\sqrt{7}\sum \frac{(i+j+k)!}{i!j!k!}(-1)^{n-i}2^i$$

where the sum is taken over all triples (i,j,k) of nonnegative integers satisfying $i+2j+3k=n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $\omega := \exp(2\pi i/7)$, then it is easy to verify that

$$(1-x_1 z)(1-x_2 z)(1-x_3 z) = 1-2z-z^2+z^3$$

where

$$x_1 := \frac{\sin(4\pi/7)}{\sin(\pi/7)} = \frac{\sin(4\pi/7)}{\sin(6\pi/7)} = \frac{\omega^2-\omega^{-2}}{\omega^3-\omega^{-3}} = 1+\omega+\omega^{-1},$$
$$x_2 := \frac{\sin(\pi/7)}{\sin(2\pi/7)} = \frac{\sin(6\pi/7)}{\sin(2\pi/7)} = \frac{\omega^3-\omega^{-3}}{\omega-\omega^{-1}} = 1+\omega^2+\omega^{-2},$$
$$x_3 := -\frac{\sin(2\pi/7)}{\sin(4\pi/7)} = \frac{\omega-\omega^{-1}}{\omega^2-\omega^{-2}} = 1+\omega^3+\omega^{-3}.$$

Hence, by the partial fraction decomposition,

$$\frac{1}{1-2z-z^2+z^3} = \frac{A_1}{1-x_1 z} + \frac{A_2}{1-x_2 z} + \frac{A_3}{1-x_3 z},$$

where

$$A_1 := \frac{1}{(1-\frac{x_2}{x_1})(1-\frac{x_3}{x_1})} = \frac{x_1^2}{2\sqrt{7}\sin(4\pi/7)},$$
$$A_2 := \frac{1}{(1-\frac{x_1}{x_2})(1-\frac{x_3}{x_2})} = -\frac{x_2^2}{2\sqrt{7}\sin(\pi/7)},$$
$$A_3 := \frac{1}{(1-\frac{x_1}{x_3})(1-\frac{x_2}{x_3})} = \frac{x_3^2}{2\sqrt{7}\sin(2\pi/7)}.$$

Finally

$$\begin{aligned}
\sum_{i+2j+3k=n} \frac{(i+j+k)!}{i!j!k!}(-1)^{n-i}2^i &= [z^n]\sum_{m=0}^{\infty}(2z+z^2-z^3)^m \\
&= [z^n]\frac{1}{1-2z-z^2+z^3} \\
&= [z^n]\left(\frac{A_1}{1-x_1 z} + \frac{A_2}{1-x_2 z} + \frac{A_3}{1-x_3 z}\right) \\
&= A_1 x_1^n + A_2 x_2^n + A_3 x_3^n \\
&= \frac{1}{2\sqrt{7}}\left(\frac{\sin^{n+1}(4\pi/7)}{\sin^{n+2}(\pi/7)} - \frac{\sin^{n+1}(\pi/7)}{\sin^{n+2}(2\pi/7)} + (-1)^n \frac{\sin^{n+1}(2\pi/7)}{\sin^{n+2}(4\pi/7)}\right)
\end{aligned}$$

and we are done. □