Problem 12116
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Proposed by R. Thaper (USA).

In a round-robin tournament with n players, each player plays every other player exactly once, and each match results in a win for one player and a loss for the other. When player A defeats player B, we call B the victim of A. At the end of the tournament, each player computes the total number of losses incurred by the player's victims. Let q be the average of this quantity over all players. Prove that there exists a player with at most $\lfloor\sqrt{q}\rfloor$ wins and a player with at most $\lfloor\sqrt{q}\rfloor$ losses.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We consider a directed graph $G=(V,E)$ where the vertices represent the players and two vertices v and w are connected by an edge from v to w if player v won against player w. Let $\text{in}(v)$ be the set of players which defeated v and let $\text{out}(v)$ be the set of players that have been defeated by v. Then the average q is given by

$$q=\frac{1}{|V|}\sum_{v\in V}\sum_{w\in\text{out}(v)}|\text{in}(w)|=\frac{1}{|V|}\sum_{w\in V}|\text{in}(w)|\sum_{v\in\text{in}(w)}1=\frac{1}{|V|}\sum_{w\in V}|\text{in}(w)|^2.$$

1) Assume that all players have more than $\lfloor\sqrt{q}\rfloor$ wins, that is $|\text{in}(w)|>\sqrt{q}$ for all $w\in V$, then

$$q=\frac{1}{|V|}\sum_{w\in V}|\text{in}(w)|^2>\frac{1}{|V|}\sum_{w\in V}q=q$$

which is a contradiction. Hence there exists a player with at most $\lfloor\sqrt{q}\rfloor$ wins.

2) Assume that all players have more than $\lfloor\sqrt{q}\rfloor$ losses, that is $|\text{out}(w)|>\sqrt{q}$ for all $w\in V$, then, since $\sum_{w\in V}|\text{in}(w)|=\sum_{w\in V}|\text{out}(w)|$, it follows by the Cauchy-Schwarz inequality that

$$|V|\cdot|V|q=\sum_{w\in V}1^2\cdot\sum_{w\in V}|\text{in}(w)|^2\geq\left(\sum_{w\in V}1\cdot|\text{in}(w)|\right)^2=\left(\sum_{w\in V}|\text{out}(w)|\right)^2>\left(\sum_{w\in V}\sqrt{q}\right)^2=|V|^2q$$

which is a contradiction. Hence there exists player with at most $\lfloor\sqrt{q}\rfloor$ losses. $\square$