Problem 12115
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Proposed by M. Drăgan (Romania).

Let a, b, c, and d be positive real numbers. Prove

$$(a^3+b^3)(a^3+c^3)(a^3+d^3)(b^3+c^3)(b^3+d^3)(c^3+d^3) \geq (a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2)^3.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show a more general inequality: if $p \geq 2$ and $a, b, c, d \geq 0$ then

$$(a^p+b^p)(a^p+c^p)(a^p+d^p)(b^p+c^p)(b^p+d^p)(c^p+d^p) \geq 2^{6-2p}(a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2)^p.$$

By the Power Mean Inequality, if $x, y \geq 0$ then $x^p + y^p \geq 2^{1-p/2}(x^2+y^2)^{p/2}$ and it follows that the left-hand side is greater or equal to

$$2^{6-3p}\left((a^2+b^2)(a^2+c^2)(a^2+d^2)(b^2+c^2)(b^2+d^2)(c^2+d^2)\right)^{p/2}.$$

Hence it remains to show that

$$(a^2+b^2)(a^2+c^2)(a^2+d^2)(b^2+c^2)(b^2+d^2)(c^2+d^2) \geq 4(a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2)^2$$

that is the generalized inequality for $p = 2$.
After expanding both sides multiplied by 3, it reduces to

$$3\sum_{\text{sym}} a^6b^4c^2d^0 + \sum_{\text{sym}} a^6b^2c^2d^2 \geq 3\sum_{\text{sym}} a^4b^4c^2d^2 + \sum_{\text{sym}} a^4b^4c^4d^0$$

which holds because, by Muirhead's inequality,

$$\sum_{\text{sym}} a^6b^4c^2d^0 \geq \sum_{\text{sym}} a^4b^4c^2d^2 \quad \text{and} \quad \sum_{\text{sym}} a^6b^2c^2d^2 \geq \sum_{\text{sym}} a^4b^4c^4d^0.$$

$\square$