Problem 12114
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Proposed by Z. Franco (USA).

Let n be a positive integer, and let $A_n = \{1/n, 2/n, \ldots, n/n\}$. Let a_n be the sum of the numerators in A_n when these fractions are expressed in lowest terms. Find $\sum_{n=1}^{\infty} a_n/n^4$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. More generally, we show that for any $s > 3$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{\zeta(s)}{2}\left(1 + \frac{\zeta(s-2)}{\zeta(s-1)}\right).$$

We have that

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{k}{\gcd(n,k)} = 1 + \frac{1}{2}\left(\sum_{k=1}^{n-1} \frac{k}{\gcd(n,k)} + \sum_{k=1}^{n-1} \frac{k}{\gcd(n,n-k)}\right) \\ &= 1 + \frac{1}{2}\left(\sum_{k=1}^{n-1} \frac{k}{\gcd(n,k)} + \sum_{k=1}^{n-1} \frac{n-k}{\gcd(n,k)}\right) = 1 + \frac{1}{2}\sum_{k=1}^{n-1} \frac{n}{\gcd(n,k)} = \frac{1+b_n}{2} \end{aligned}$$

where

$$b_n = \sum_{k=1}^{n} \frac{n}{\gcd(n,k)} = \sum_{d|n} \frac{n\varphi(n/d)}{d} = \sum_{d|n} d\varphi(d)$$

is multiplicative . Since for any prime p and for any positive integer r,

$$b_{p^r} = 1 + \sum_{k=1}^{r} p^k \varphi(p^k) = 1 + \sum_{k=1}^{r} (p^{2k} - p^{2k-1}) = \frac{p^{2r+1}+1}{p+1},$$

it follows that for $s > 3$, the Dirichlet series of the multiplicative function b_n reduces to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n}{n^s} &= \prod_p \left(1 + \sum_{r=1}^{\infty} \frac{b_{p^r}}{p^{rs}}\right) = \prod_p \left(1 + \frac{p}{p+1}\sum_{r=1}^{\infty} \frac{1}{p^{(s-2)r}} + \frac{1}{p+1}\sum_{r=1}^{\infty} \frac{1}{p^{sr}}\right) \\ &= \prod_p \left(1 + \frac{p}{p+1} \cdot \frac{1}{p^{s-2}-1} + \frac{1}{p+1} \cdot \frac{1}{p^s - 1}\right) \\ &= \prod_p \frac{\left(1 - \frac{1}{p^{s-1}}\right)}{\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-2}}\right)} = \frac{\zeta(s)\zeta(s-2)}{\zeta(s-1)}. \end{aligned}$$

Therefore, we may conclude that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{b_n}{n^s}\right) = \frac{\zeta(s)}{2}\left(1 + \frac{\zeta(s-2)}{\zeta(s-1)}\right).$$

□