Problem 12113
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Proposed by R. P. Stanley (USA).

Define $f(n)$ and $g(n)$ for $n \geq 0$ by

$$\sum_{n\geq 0} f(n)x^n = \sum_{j\geq 0} x^{2^j} \prod_{k=0}^{j-1} \left(1 + x^{2^k} + x^{3\cdot 2^k}\right)$$

and

$$\sum_{n\geq 0} g(n)x^n = \prod_{i\geq 0} \left(1 + x^{2^i} + x^{3\cdot 2^i}\right).$$

Find all values of n for which $f(n) = g(n)$, and find $f(n)$ for these values.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$F(x) := \sum_{n\geq 0} f(n)x^n = x + (1 + x + x^3)\, F(x^2)$$
$$G(x) := \sum_{n\geq 0} g(n)x^n = (1 + x + x^3)\, G(x^2)$$

and therefore the sequences $f(n)$ and $g(n)$ satisfy the same recurrence relation: $x(1) = 1$ and for $n \geq 2$,

$$x(n) = \begin{cases} x(n/2) & \text{if } n \text{ is even,} \\ x(n-1) + x(n-3) & \text{if } n \text{ is odd} \end{cases}$$

but have different initial conditions, i.e. $f(0) = 0$ and $g(0) = 1$.
It follows that $f(n) \leq g(n)$, and equality holds if and only if n is 1 or it is of the form $2^i(3 \cdot 2^j - 1)$ with $i, j \geq 0$. Moreover

$$f(n) = g(n) = \begin{cases} 1 & \text{if } n = 1, \\ F_{j+2} & \text{if } n = 2^i(3 \cdot 2^j - 1) \text{ with } i, j \geq 0, \end{cases}$$

where F_k is the k-th Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ with $k \geq 2$).
Indeed, if $j = 0$ then

$$f(2^i(3 \cdot 2^0 - 1)) = f(2^{i+1}) = f(1) = 1 = F_2.$$

If $j = 1$ then

$$f(2^i(3 \cdot 2^1 - 1)) = f(2^i \cdot 5) = f(5) = f(4) + f(2) = 2f(1) = 2 = F_3.$$

Finally, for $j \geq 2$,

$$\begin{aligned} f(2^i \cdot (3 \cdot 2^j - 1)) &= f(3 \cdot 2^j - 1) = f(3 \cdot 2^j - 2) + f(3 \cdot 2^j - 4) \\ &= f(3 \cdot 2^{j-1} - 1) + f(3 \cdot 2^{j-2} - 1) = F_{j+1} + F_j = F_{j+2} \end{aligned}$$

and we may conclude that the above formula holds by induction. □