Problem 12110
(American Mathematical Monthly, Vol.126, April 2019)

Proposed by P. J. Rodriguez de Rivera and A. Plaza (Spain).

Let $\alpha_k = (k + \sqrt{k^2+4})/2$. *Evaluate*

$$\lim_{k\to\infty} \prod_{n=1}^{\infty} \left(1 - \frac{k}{\alpha_k^n + \alpha_k}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\alpha_k^2 = \frac{k^2 + k^2 + 4 + 2k\sqrt{k^2+4}}{4} = k\alpha_k + 1.$$

Therefore

$$1 - \frac{k}{\alpha_k^n + \alpha_k} = \frac{\alpha_k^{n+1} + \alpha_k^2 - k\alpha_k}{\alpha_k^{n+1} + \alpha_k^2} = \frac{\alpha_k^{n+1} + 1}{\alpha_k^{n+1} + \alpha_k^2} = \frac{1 + \alpha_k^{-(n+1)}}{1 + \alpha_k^{-(n-1)}}.$$

Finally, since $\alpha_k > 1$ for $k > 0$, and $\lim_{k\to\infty} \alpha_k^{-1} = 0$, we may conclude that

$$\begin{aligned}
\lim_{k\to\infty} \prod_{n=1}^{\infty} \left(1 - \frac{k}{\alpha_k^n + \alpha_k}\right) &= \lim_{k\to\infty} \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1 + \alpha_k^{-(n+1)}}{1 + \alpha_k^{-(n-1)}} \\
&= \lim_{k\to\infty} \lim_{N\to\infty} \frac{(1 + \alpha_k^{-N})(1 + \alpha_k^{-(N+1)})}{(1 + \alpha_k^{0})(1 + \alpha_k^{-1})} \\
&= \lim_{k\to\infty} \frac{1}{2(1 + \alpha_k^{-1})} = \frac{1}{2}.
\end{aligned}$$

□