Problem 12107
(American Mathematical Monthly, Vol.126, April 2019)

Proposed by C. I. Vălean (Romania).

Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)}\,dx\,dy = G$$

where G is the Catalan's constant $\sum_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)} &= \int_0^{\pi/4}\left(\int_0^{\pi/4} \frac{\cos(u)\cos(v)}{\cos^2(u)-\sin^2(v)}\,dv\right)du \quad (x=\tan(u), y=\tan(v))\\
&= \frac{1}{2}\int_0^{\pi/4}\left[\log\left(\frac{\cos(u)+\sin(v)}{\cos(u)-\sin(v)}\right)\right]_{v=0}^{\pi/4} du\\
&= \frac{1}{2}\int_0^{\pi/4}\log\left(\frac{\cos(u)+\frac{1}{\sqrt{2}}}{\cos(u)-\frac{1}{\sqrt{2}}}\right)du = \frac{1}{4}\int_{-\pi/4}^{\pi/4}\log\left(\frac{\cos(u)+\frac{1}{\sqrt{2}}}{\cos(u)-\frac{1}{\sqrt{2}}}\right)du\\
&= \frac{1}{4}\int_0^{\pi/2}\log\left(\frac{\cos(s)+\sin(s)+1}{\cos(s)+\sin(s)-1}\right)ds \qquad \left(u = s-\frac{\pi}{4}\right)\\
&= \frac{1}{2}\int_0^1 \frac{\log\left(\frac{1+t}{t(1-t)}\right)}{1+t^2}\,dt \qquad (t=\tan(s/2))\\
&= -\frac{1}{2}\int_0^1 \frac{\log(t)}{1+t^2}\,dt - \frac{1}{2}\int_0^1 \frac{\log\left(\frac{1-t}{1+t}\right)}{1+t^2}\,dt\\
&= -\frac{1}{2}\int_0^1 \frac{\log(t)}{1+t^2}\,dt - \frac{1}{2}\int_0^1 \frac{\log(r)}{1+r^2}\,dr \qquad \left(r=\frac{1-t}{1+t}\right)\\
&= -\int_0^1 \frac{\log(t)}{1+t^2}\,dt = [-\log(t)\arctan(t)]_0^1 + \int_0^1 \frac{\arctan(t)}{t}\,dt\\
&= \int_0^1 \sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^{2n-2}}{2n-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 t^{2n-2}\,dt\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} = G
\end{aligned}$$

and we are done. □