Problem 12106
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For any positive integer n, prove

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{1\leq i\leq j\leq k}\frac{1}{ij}=\sum_{1\leq i\leq j\leq n}\frac{2^n-2^{n-i}}{ij}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$A_n:=\sum_{k=1}^{n}\binom{n}{k}\sum_{1\leq i\leq j\leq k}\frac{1}{ij}=\sum_{k=1}^{n}\binom{n}{k}\sum_{j=1}^{k}\frac{H_j}{j}\quad\text{and}\quad B_n:=\sum_{1\leq i\leq j\leq n}\frac{2^n-2^{n-i}}{ij}.$$

where $H_j=\sum_{i=1}^{j}\frac{1}{i}$. Then $A_1=B_1=1$. Moreover, for $n>1$,

$$\begin{aligned}A_n&=\sum_{k=1}^{n}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)\sum_{j=1}^{k}\frac{H_j}{j}=A_{n-1}+\sum_{k=1}^{n}\binom{n-1}{k-1}\sum_{j=1}^{k-1}\frac{H_j}{j}+\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{H_k}{k}\\&=2A_{n-1}+\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}H_k\end{aligned}$$

and

$$B_n=2\sum_{1\leq i\leq j\leq n-1}\frac{2^{n-1}-2^{n-1-i}}{ij}+\sum_{i=1}^{n}\frac{2^n-2^{n-i}}{in}=2B_{n-1}+\frac{1}{n}\sum_{i=1}^{n}\frac{2^n-2^{n-i}}{i}.$$

Hence, by induction, for any positive integer n, $A_n=B_n$ if and only if $a_n=b_n$ where

$$a_n:=\sum_{k=1}^{n}\binom{n}{k}H_k\quad\text{and}\quad b_n:=\sum_{i=1}^{n}\frac{2^n-2^{n-i}}{i}.$$

We have that $a_1=b_1=1$. Moreover, for $n>1$,

$$\begin{aligned}a_n&=\sum_{k=1}^{n}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)H_k=a_{n-1}+\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}+\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{1}{k}\\&=2a_{n-1}+\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}=2a_{n-1}+\frac{2^n-1}{n}\end{aligned}$$

and

$$b_n=2\sum_{i=1}^{n-1}\frac{2^{n-1}-2^{n-1-i}}{i}+\frac{2^n-2^{n-n}}{n}=2b_{n-1}+\frac{2^n-1}{n}.$$

Therefore, again by induction, for any positive integer n, $a_n=b_n$ and we are done. □