Problem 12105
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Proposed by G. Brookfield (USA).

Let r be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r$. Suppose that f has real roots a, b, and c. Prove $a, b, c \in [-1, 1]$ and

$$|\arcsin(a)| + |\arcsin(b)| + |\arcsin(c)| = \pi.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. If $x \in \{-1, 0, 1\}$ and $f(x) = 0$ then $r = 0$, $\{a, b, c\} = \{-1, 0, 1\}$ and the desired arcsin identity is valid.
We assume now that $x \notin \{-1, 0, 1\}$ and therefore $r \neq 0$. If $f(x) = 0$ for some $r \in \mathbb{R} \setminus \{0\}$ then

$$f(x) = xr^2 + 2(x^2 - 1)r + x^3 - x = 0$$

is a quadratic equation with respect to r and it follows that

$$\frac{\Delta}{4} = (x^2 - 1)^2 - x(x^3 - x) = 1 - x^2 \geq 0$$

which implies that $|x| \in (0, 1)$. It follows that the real roots a, b, c are such that $|a|, |b|, |c| \in (0, 1)$. Moreover,

$$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = -f(x)f(-x) = x^6 - 2(r^2 + 1)x^4 + (r^4 + 6r^2 + 1)x^2 - 4r^2$$

and therefore

$$a^2 + b^2 + c^2 = 2(r^2 + 2), \quad a^2b^2 + b^2c^2 + c^2a^2 = (r^4 + 6r^2 + 1), \quad a^2b^2c^2 = 4r^2.$$

Hence

$$(a^2 + b^2 + c^2)^2 + 4a^2b^2c^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) = (2(r^2 + 1))^2 + 4 \cdot 4r^2 - 4 \cdot (r^4 + 6r^2 + 1) = 0 \quad (*).$$

Finally we show that desired arcsin identity is implied by $(*)$.
Note that since $a + b + c = -2r = -abc$ and $abc = 2r$ it follows that $c = -(a + b)/(1 + ab)$ and for $0 < |a|, |b| < 1$,

$$\arcsin|a| + \arcsin|b| > \arcsin|c| = \arcsin\left|\frac{a + b}{1 + ab}\right|$$

because $h(t) = \arcsin(\tanh(t))$ is monotone and strictly subadditive in $(0, +\infty)$ ($h(0) = 0$ and h is strictly concave in $[0, +\infty)$).
We may assume that $0 < |a| \leq |b| \leq |c| < 1$. Thus, by $(*)$,

$$\begin{aligned}
(a^2 + b^2 + c^2)^2 + &4a^2b^2c^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) = 0 \\
&\Leftrightarrow a^4 + b^4 + c^4 + 4a^2b^2c^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) \\
&\Leftrightarrow 4a^2b^2(1 - a^2)(1 - b^2) = (c^2 - a^2(1 - b^2) - b^2(1 - a^2))^2 \\
&\Leftrightarrow a^2(1 - b^2) + b^2(1 - a^2) \pm 2|a||b|\sqrt{1 - a^2}\sqrt{1 - b^2} = c^2 \\
&\Leftrightarrow |a|\sqrt{1 - b^2} + |b|\sqrt{1 - a^2} = |c| \quad \vee \quad \left||a|\sqrt{1 - b^2} - |b|\sqrt{1 - a^2}\right| = |c| \\
&\Rightarrow |a|\sqrt{1 - b^2} + |b|\sqrt{1 - a^2} = |c| \qquad (0 < |a| \leq |b| \leq |c| < 1) \\
&\Leftrightarrow \sin(\arcsin|a| + \arcsin|b|) = \sin(\arcsin|c|) \\
&\Leftrightarrow \arcsin|a| + \arcsin|b| = \arcsin|c| \quad \vee \quad \arcsin|a| + \arcsin|b| = \pi - \arcsin|c| \\
&\Rightarrow |\arcsin(a)| + |\arcsin(b)| + |\arcsin(c)| = \pi \qquad (\arcsin|a| + \arcsin|b| > \arcsin|c|)
\end{aligned}$$

and we are done.

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